

Language: **Bosnian**

Subota, 28.04.2012.

Zadatak 1. Neka su A, B i C točke kružnice Γ sa centrom O , takve da je $\angle ABC > 90^\circ$. Neka je D točka presjeka prave AB i prave okomite na AC u točki C . Neka je ℓ prava koja sadrži točku D i okomita je na pravu AO . Dalje, neka je E točka presjeka pravih ℓ i AC , a F ona točka presjeka kružnice Γ i prave ℓ koja se nalazi između točaka D i E .

Dokazati da se opisane kružnice trokutova BFE i CFD dodiruju u točki F .

Zadatak 2. Dokazati da nejednakost

$$\sum_{\text{cyc}} (x+y) \sqrt{(z+x)(z+y)} \geq 4(xy + yz + zx)$$

važi za sve pozitivne realne brojeve x, y i z .

Suma na lijevoj strani gornje nejednakosti jednaka je

$$(x+y) \sqrt{(z+x)(z+y)} + (y+z) \sqrt{(x+y)(x+z)} + (z+x) \sqrt{(y+z)(y+x)}.$$

Zadatak 3. Za prirodan broj n neka je $P_n = \{2^n, 2^{n-1} \cdot 3, 2^{n-2} \cdot 3^2, \dots, 3^n\}$. Za svaki podskup X skupa P_n označimo sa S_X zbir svih elemenata skupa X , pri čemu je $S_\emptyset = 0$, gdje je $\emptyset$ prazan skup. Neka je y realan broj takav da je $0 \leq y \leq 3^{n+1} - 2^{n+1}$.

Dokazati da postoji podskup Y skupa P_n za koji je $0 \leq y - S_Y < 2^n$.

Zadatak 4. Odrediti sve funkcije $f : \mathbb{N} \rightarrow \mathbb{N}$ koje zadovoljavaju sljedeća dva uvjeta:

- (i) $f(n!) = f(n)!$, za svaki prirodan broj n ,
- (ii) $m - n$ dijeli $f(m) - f(n)$, za sve različite prirodne brojeve m i n .

*Svaki zadatak vrijedi 10 poena.
Vrijeme za rad: 4 sata i 30 minuta.*



Problem 2.

Solution 1. We will obtain the inequality by adding the inequalities

$$(x+y)\sqrt{(z+x)(z+y)} \geq 2xy + yz + zx$$

for cyclic permutation of x, y, z .

Squaring both sides of this inequality we obtain

$$(x+y)^2(z+x)(z+y) \geq 4x^2y^2 + y^2z^2 + z^2x^2 + 4xy^2z + 4x^2yz + 2xyz^2$$

which is equivalent to

$$x^3y + xy^3 + z(x^3 + y^3) \geq 2x^2y^2 + xyz(x+y)$$

which can be rearranged to

$$(xy + yz + zx)(x - y)^2 \geq 0,$$

which is clearly true.

Solution 2. For positive real numbers x, y, z there exists a triangle with the side lengths $\sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ and the area $K = \sqrt{xy+yz+zx}/2$.

The existence of the triangle is clear by simple checking of the triangle inequality. To prove the area formula, we have

$$K = \frac{1}{2}\sqrt{x+y}\sqrt{z+x}\sin\alpha,$$

where α is the angle between the sides of length $\sqrt{x+y}$ and $\sqrt{z+x}$. On the other hand, from the law of cosines we have

$$\cos\alpha = \frac{x+y+z+x-y-z}{2\sqrt{(x+y)(z+x)}} = \frac{x}{\sqrt{(x+y)(z+x)}}$$

and

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \frac{\sqrt{xy+yz+zx}}{\sqrt{(x+y)(z+x)}}.$$

Now the inequality is equivalent to

$$\sqrt{x+y}\sqrt{y+z}\sqrt{z+x} \sum_{cyc} \sqrt{x+y} \geq 16K^2.$$

This can be rewritten as

$$\frac{\sqrt{x+y}\sqrt{y+z}\sqrt{z+x}}{4K} \geq 2 \frac{K}{\sum_{cyc} \sqrt{x+y}/2}$$

to become the Euler inequality $R \geq 2r$.



Problem 3.

Solution 1. Let $\alpha = 3/2$ so $1 + \alpha > \alpha^2$.

Given y , we construct Y algorithmically. Let $Y = \emptyset$ and of course $S_\emptyset = 0$. For $i = 0$ to m , perform the following operation:

$$\text{If } S_Y + 2^i 3^{m-i} \leq y, \text{ then replace } Y \text{ by } Y \cup \{2^i 3^{m-i}\}.$$

When this process is finished, we have a subset Y of P_m such that $S_Y \leq y$.

Notice that the elements of P_m are in ascending order of size as given, and may alternatively be described as $2^m, 2^m \alpha, 2^m \alpha^2, \dots, 2^m \alpha^m$. If any member of this list is not in Y , then no two consecutive members of the list to the left of the omitted member can both be in Y . This is because $1 + \alpha > \alpha^2$, and the greedy nature of the process used to construct Y .

Therefore either $Y = P_m$, in which case $y = 3^{m+1} - 2^{m+1}$ and all is well, or at least one of the two leftmost elements of the list is omitted from Y .

If 2^m is not omitted from Y , then the algorithmic process ensures that $(S_Y - 2^m) + 2^{m-1} 3 > y$, and so $y - S_Y < 2^m$. On the other hand, if 2^m is omitted from Y , then $y - S_Y < 2^m$.

Solution 2. Note that $3^{m+1} - 2^{m+1} = (3 - 2)(3^m + 3^{m-1} \cdot 2 + \dots + 3 \cdot 2^{m-1} + 2^m) = S_{P_m}$. Dividing every element of P_m by 2^m gives us the following equivalent problem:

Let m be a positive integer, $a = 3/2$, and $Q_m = \{1, a, a^2, \dots, a^m\}$. Show that for any real number x satisfying $0 \leq x \leq 1 + a + a^2 + \dots + a^m$, there exists a subset X of Q_m such that $0 \leq x - S_X < 1$.

We will prove this problem by induction on m . When $m = 1$, $S_\emptyset = 0$, $S_{\{1\}} = 1$, $S_{\{a\}} = 3/2$, $S_{\{1,a\}} = 5/2$. Since the difference between any two consecutive of them is at most 1, the claim is true.

Suppose that the statement is true for positive integer m . Let x be a real number with $0 \leq x \leq 1 + a + a^2 + \dots + a^{m+1}$. If $0 \leq x \leq 1 + a + a^2 + \dots + a^m$, then by the induction hypothesis there exists a subset X of $Q_m \subset Q_{m+1}$ such that $0 \leq x - S_X < 1$.

If $\frac{a^{m+1} - 1}{a - 1} = 1 + a + a^2 + \dots + a^m < x$, then $x > a^{m+1}$ as

$$\frac{a^{m+1} - 1}{a - 1} = 2(a^{m+1} - 1) = a^{m+1} + (a^{m+1} - 2) \geq a^{m+1} + a^2 - 2 = a^{m+1} + \frac{1}{4}.$$

Therefore $0 < (x - a^{m+1}) \leq 1 + a + a^2 + \dots + a^m$. Again by the induction hypothesis, there exists a subset X of Q_m satisfying $0 \leq (x - a^{m+1}) - S_X < 1$. Hence $0 \leq x - S_{X'} < 1$ where $X' = X \cup \{a^{m+1}\} \subset Q_{m+1}$.



Problem 4.

Solution 1. There are three such functions: the constant functions 1, 2 and the identity function $\text{id}_{\mathbf{Z}^+}$. These functions clearly satisfy the conditions in the hypothesis. Let us prove that there are only ones.

Consider such a function f and suppose that it has a fixed point $a \geq 3$, that is $f(a) = a$. Then $a!, (a!)!, \dots$ are all fixed points of f , hence the function f has a strictly increasing sequence $a_1 < a_2 < \dots < a_k < \dots$ of fixed points. For a positive integer n , $a_k - n$ divides $a_k - f(n) = f(a_k) - f(n)$ for every $k \in \mathbf{Z}^+$. Also $a_k - n$ divides $a_k - n$, so it divides $a_k - f(n) - (a_k - n) = n - f(n)$. This is possible only if $f(n) = n$, hence in this case we get $f = \text{id}_{\mathbf{Z}^+}$.

Now suppose that f has no fixed points greater than 2. Let $p \geq 5$ be a prime and notice that by Wilson's Theorem we have $(p-2)! \equiv 1 \pmod{p}$. Therefore p divides $(p-2)! - 1$. But $(p-2)! - 1$ divides $f((p-2)!) - f(1)$, hence p divides $f((p-2)!) - f(1) = (f(p-2))! - f(1)$. Clearly we have $f(1) = 1$ or $f(1) = 2$. As $p \geq 5$, the fact that p divides $(f(p-2))! - f(1)$ implies that $f(p-2) < p$. It is easy to check, again by Wilson's Theorem, that p does not divide $(p-1)! - 1$ and $(p-1)! - 2$, hence we deduce that $f(p-2) \leq p-2$. On the other hand, $p-3 = (p-2) - 1$ divides $f(p-2) - f(1) \leq (p-2) - 1$. Thus either $f(p-2) = f(1)$ or $f(p-2) = p-2$. As $p-2 \geq 3$, the last case is excluded, since the function f has no fixed points greater than 2. It follows $f(p-2) = f(1)$ and this property holds for all primes $p \geq 5$. Taking n any positive integer, we deduce that $p-2-n$ divides $f(p-2) - f(n) = f(1) - f(n)$ for all primes $p \geq 5$. Thus $f(n) = f(1)$, hence f is the constant function 1 or 2.

Solution 2. Note first that if $f(n_0) = n_0$, then $m - n_0 \mid f(m) - m$ for all $m \in \mathbf{Z}^+$. If $f(n_0) = n_0$ for infinitely many $n_0 \in \mathbf{Z}^+$, then $f(m) - m$ has infinitely many divisors, hence $f(m) = m$ for all $m \in \mathbf{Z}^+$. On the other hand, if $f(n_0) = n_0$ for some $n_0 \geq 3$, then f fixes each term of the sequence $(n_k)_{k=0}^\infty$, which is recursively defined by $n_k = n_{k-1}!$. Hence if $f(3) = 3$, then $f(n) = n$ for all $n \in \mathbf{Z}^+$.

We may assume that $f(3) \neq 3$. Since $f(1) = f(1)!$, and $f(2) = f(2)!$, $f(1), f(2) \in \{1, 2\}$. We have $4 = 3! - 2 \mid f(3)! - f(2)$. This together with $f(3) \neq 3$ implies that $f(3) \in \{1, 2\}$. Let $n > 3$, then $n! - 3 \mid f(n)! - f(3)$ and $3 \nmid f(n)!$, i.e. $f(n)! \in \{1, 2\}$. Hence we conclude that $f(n) \in \{1, 2\}$ for all $n \in \mathbf{Z}^+$.

If f is not constant, then there exist positive integers m, n with $\{f(n), f(m)\} = \{1, 2\}$. Let $k = 2 + \max\{m, n\}$. If $f(k) \neq f(m)$, then $k - m \mid f(k) - f(m)$. This is a contradiction as $|f(k) - f(m)| = 1$ and $k - m \geq 2$.

Therefore the functions satisfying the conditions are $f \equiv 1, f \equiv 2, f = \text{id}_{\mathbf{Z}^+}$.



OFFICIAL RESULTS OF BMO 2012

Rank	Code	Name	Surname	P1	P2	P3	P4	Σ	Medal
1	BIH3	Harun	HINDIJA	10	10	10	10	40	GOLD
1	HEL5	Alexandros	MOUSATOV	10	10	10	10	40	GOLD
1	ROU3	Octav	DRĂGOI	10	10	10	10	40	GOLD
1	SRB1	Teodor von	BURG	10	10	10	10	40	GOLD
1	SRB3	Dušan	ŠOBOT	10	10	10	10	40	GOLD
1	SRB4	Igor	SPASOJEVIĆ	10	10	10	10	40	GOLD
1	TUR2	Yunus Emre	DEMİRCİ	10	10	10	10	40	GOLD
1	TUR4	Mehmet	SÖNMEZ	10	10	10	10	40	GOLD
1	TUR5	Berfin	ŞİMŞEK	10	10	10	10	40	GOLD
10	HEL3	Panagiotis	LOLAS	10	10	10	9	39	GOLD
10	ROU2	Ömer	CERRAHOĞLU	10	10	9	10	39	GOLD
10	ROU5	Ştefan	SPĂTARU	10	10	10	9	39	GOLD
10	TUR3	Ufuk	KANAT	10	10	9	10	39	GOLD
10	TURB4	Muhammed İkbāl	ULVİ	10	10	9	10	39	GOLD
15	BGR5	Yordan	YORDANOV	10	10	9	9	38	SILVER
15	SAU6	Al Yazeed	PASUNI	10	10	8	10	38	SILVER
15	SRB5	Rade	ŠPEGAR	10	10	10	8	38	SILVER
15	TURB5	Burak	VARICI	10	10	10	8	38	SILVER
19	ROU1	Hai	TRAN BACH	10	10	10	7	37	SILVER
20	TUR6	Mehmet Akif	YILDIZ	10	10	6	10	36	SILVER
21	HEL6	Zacharias	TSAMPASIDIS	10	10	10	4	34	SILVER
22	FRA1	Arthur	BLANC-RENAUDIE	10	10	10	3	33	SILVER
22	ROU4	Ştefan	IVANOVICI	10	10	9	4	33	SILVER
22	TURB2	Eren	DURLANIK	10	10	9	4	33	SILVER
22	UNK5	Matei	MANDACHE	10	10	9	4	33	SILVER
26	BGR1	Ivailo	HARTARSKI	10	10	9	3	32	SILVER
26	ITA1	Gioacchino	ANTONELLI	10	10	9	3	32	SILVER
26	TURB6	Orhan Tahir	YAVAŞCAN	10	10	10	2	32	SILVER
29	BGR6	Bogdan	STANKOV	10	9	8	4	31	SILVER
29	BIH2	Ratko	DARDA	10	10	1	10	31	SILVER
29	IDN2	Christa Lorenzia	SOESANTO	10	10	10	1	31	SILVER
29	ITA2	Dario	ASCARI	10	10	10	1	31	SILVER
29	TUR1	Mehmet Efe	AKENGİN	10	10	1	10	31	SILVER
29	TURB3	Fatih	KALEOĞLU	10	10	10	1	31	SILVER
35	FRA4	Seginus	MOWLAVI	10	10	10	0	30	SILVER
35	FRA5	Matthieu	PIQUEREZ	10	7	10	3	30	SILVER
35	ITA3	Luigi	PAGANO	10	10	9	1	30	SILVER
35	ITA5	Daniele	SEMOLA	10	10	10	0	30	SILVER
35	KAZ1	Saken	DUSHAYEV	10	10	10	0	30	SILVER
35	KAZ5	Alibek	SAILANBAYEV	10	10	0	10	30	SILVER
35	TKM1	Palvan	AGAMYRADOV	10	10	10	0	30	SILVER
35	UNK3	Daniel	HU	10	0	10	10	30	SILVER
43	BIH4	Hamza	MERZIC	10	10	0	8	28	BRONZE
43	ITA6	Marco	TREVISIOL	5	10	9	4	28	BRONZE
45	KAZ3	Sergazy	KALMURZAYEV	10	10	1	6	27	BRONZE
45	TJK1	Muhammadfiruz	HASANOV	6	10	10	1	27	BRONZE
45	UNK1	Robin	ELLIOTT	8	10	8	1	27	BRONZE

48	MDA5	Andrei	IVANOV	10	10	5	0	25	BRONZE
48	ROU6	Florina	TOMA	10	10	4	1	25	BRONZE
50	FRA2	Sébastien	CHEVALEYRE	6	10	8	0	24	BRONZE
50	KAZ4	Yerniyaz	NURGABYLOV	10	4	0	10	24	BRONZE
50	TJK6	Saidmuhammadi	SHODAVLAT	10	10	1	3	24	BRONZE
53	BGR4	Pavlena	NENOVA	10	10	3	0	23	BRONZE
53	FRA6	Victor	QUACH	4	10	9	0	23	BRONZE
53	KAZ6	Temirlan	SEILOV	10	10	0	3	23	BRONZE
53	MDA2	Cristian	CERNEANU	10	3	10	0	23	BRONZE
53	MKD4	Filip	STANKOVSKI	10	10	1	2	23	BRONZE
53	TJK3	Elmurod	KHUSRAVI	10	10	0	3	23	BRONZE
53	TURB1	Eray	AYDIN	10	10	0	3	23	BRONZE
60	BGR2	Radoslav	KAFOV	10	10	1	1	22	BRONZE
60	IDN1	Bivan Alzacky	HARMANTO	10	10	1	1	22	BRONZE
60	KAZ2	Akzhol	IBRAIMOV	10	10	2	0	22	BRONZE
60	MDA3	Mihai	INDRICEAN	10	10	1	1	22	BRONZE
60	TJK4	Musokhon	MUQAYUMKHONOV	10	10	1	1	22	BRONZE
65	FRA3	Louise	GASSOT	10	0	10	1	21	BRONZE
65	ITA4	Francesco	SALA	10	10	0	1	21	BRONZE
65	MDA1	Cristian	ZANOCI	10	10	1	0	21	BRONZE
65	MDA4	Dinis	CHEIAN	10	10	1	0	21	BRONZE
65	MDA6	Dionisie	NIPOMICI	10	10	1	0	21	BRONZE
65	MKD1	Rosica	DEJANOVSKA	10	10	1	0	21	BRONZE
65	SRB2	Ivan	DAMNJANOVIĆ	0	10	10	1	21	BRONZE
65	TJK2	Abdulvosit	ISMAILOV	10	10	0	1	21	BRONZE
65	TJK5	Sadi	OKILNAZARZODA	10	10	0	1	21	BRONZE
65	TKM5	Ashyrmuhammet	SERDAROV	10	10	1	0	21	BRONZE
75	ALB3	Erjona	TOPALLI	10	10	0	0	20	BRONZE
75	AZE1	Izmir	ALIYEV	10	10	0	0	20	BRONZE
75	CYP5	Thomas	THOMA	10	10	0	0	20	BRONZE
75	HEL1	Panagiotis	DIMAKIS	10	10	0	0	20	BRONZE
75	MKD2	Vasil	KUZEVSKI	10	10	0	0	20	BRONZE
75	MKD5	Stefan	STOJCHEVSKI	10	9	1	0	20	BRONZE
75	SAU1	Saleh	ALGAMDI	10	10	0	0	20	BRONZE
75	SAU2	Abdulahman	ALHARBI	10	10	0	0	20	BRONZE
75	SAU3	Doha	ALJEDDAWI	10	10	0	0	20	BRONZE
75	SAU4	Hasan	EID	10	10	0	0	20	BRONZE
75	SRB6	Ivan	TANASIJEVIĆ	10	10	0	0	20	BRONZE
75	TKM2	Shanur	ALLABAYEV	10	9	0	1	20	BRONZE
75	UNK2	Gabriel	GENDLER	1	9	10	0	20	BRONZE
88	CYP4	Anastasios	STYLIANOU	8	10	0	1	19	HM
89	BIH1	Vladimir	IVKOVIC	10	4	0	4	18	HM
90	BGR3	Radoslav	KOMITOV	10	6	0	0	16	HM
91	CYP6	Marios	VOSKOU	0	10	4	1	15	HM
91	HEL2	Iason	KOUNTOURIDIS	4	10	1	0	15	HM
91	IDN5	Christopher	WIRIAWAN	4	10	0	1	15	HM
91	TKM4	Nazar	RAHMANOV	10	4	1	0	15	HM
91	UNK4	Matthew	JASPER	4	0	10	1	15	HM
96	AZE3	Suleyman	HASANZADE	10	4	0	0	14	HM
96	AZE6	Gunduz	SALIMLI	10	4	0	0	14	HM

96	MKD3	Aleksandar	MOMIROVSKI	4	10	0	0	14	HM
96	UNK6	Harry	METREBIAN	10	4	0	0	14	HM
100	MNE2	Olja	KRSTOVIĆ	0	10	2	1	13	HM
100	SAU5	Husain	EID	9	4	0	0	13	
102	AZE2	Hasan	HASANZADE	10	1	0	1	12	HM
102	AZE4	Shahin	HUSEYNLI	10	2	0	0	12	HM
102	BIH5	Sead	DELALIC	2	10	0	0	12	HM
102	CYP2	Georgios-Rafael	PATSALIDIS	10	0	1	1	12	HM
106	ALB6	Genti	GJIKI	1	10	0	0	11	HM
106	AZE5	Farhad	SAFARLI	10	1	0	0	11	HM
106	BIH6	Ivan	BARTULOVIC	1	10	0	0	11	HM
106	MKD6	Marija	TEPEGJOZOVA	3	8	0	0	11	
106	TKM3	Fahruddin	BARATOV	1	10	0	0	11	HM
111	TKM6	Merdan	SOLTANOV	10	0	0	0	10	HM
112	CYP3	Andreas	SOTIRIOU	4	5	0	0	9	
113	ALB4	Gledis	KALLÇO	2	4	1	0	7	
114	HEL4	Eleftherios	MALLIOS	6	0	0	0	6	
115	CYP1	Dimitris	MOUYIS	0	3	1	0	4	
116	MNE1	Oleg	CMILJANIĆ	1	2	0	0	3	
117	ALB5	Florida	AHMETAJ	1	0	0	1	2	
118	ALB1	Boriana	GJURA	0	1	0	0	1	
119	AFG1	Abdul Qayoum	SAMADI	0	0	0	0	0	
119	AFG2	Mohammad Omar	HAMIDZAI	0	0	0	0	0	
119	AFG3	Sayed Edris	HOSAINI	0	0	0	0	0	
119	AFG4	Ali Homayoun	SADEGZADA	0	0	0	0	0	
119	ALB2	Fatjon	GERA	0	0	0	0	0	