24. Балканска математичка олимпијада

Родос, Грчка – 28. април 2007

1. У конвексном четвороуглу $ABCD$ важи $AB = BC = CD$, дијагонале AC и BD су различите дужине и секу се у тачки E. Доказати да је $AE = DE$ ако и само ако је $\angle BAD + \angle ADC = 120^\circ$. *(Албанија)*

2. Наћи све функције $f : \mathbb{R} \to \mathbb{R}$ такве да за све реалне бројеве x, y важи
$$f(f(x) + y) = f(f(x) - y) + 4f(x)y.$$
(Бугарска)

3. Наћи све природне бројеве n за које постоји пермутација σ бројева $1, 2, \ldots, n$ таква да је број
$$\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{\cdots + \sqrt{\sigma(n)}}}}$$
рационалан. *(Србија)*

4. Дат је цео број $n \geq 3$. Нека су $\mathcal{C}_1, \mathcal{C}_2$ и $\mathcal{C}_3$ границе три конвексна n-тоугла у равни таква да је пресек сваке две од њих коначан скуп тачака. Наћи највећи могући број тачака скупа $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$. *(Турска)*

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Rhodes, Hellas, 28 April 2007

Problem 1.
Let $ABCD$ be a convex quadrilateral with $AB = BC = CD, AC \neq BD$ and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120^\circ$.

Problem 2.
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(f\left(x\right)+y\right) = f\left(f\left(x\right)-y\right)+4f\left(x\right)y, \text{ for any } x, y \in \mathbb{R}.$$

Problem 3.
Find all positive integers n such that there is a permutation σ of the set {1,2,…,n} for which $\sqrt{\sigma(1)+\sqrt{\sigma(2)+\sqrt{\cdots+\sqrt{\sigma(n)}}}}$ is a rational number.

***Note**: A permutation of the set {1,2,…,n} is a one-to-one function of this set to itself.*

Problem 4.
For a given positive integer $n > 2$, let C_1, C_2, C_3 be the boundaries of three convex n-gons in the plane such that $C_1 \cap C_2, C_2 \cap C_3, C_3 \cap C_1$ are finite. Find the maximum number of points of the set $C_1 \cap C_2 \cap C_3$.

Time allowed 4 hours and 30 minutes
Each problem is worth 10 points.

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Rhodes, Hellas (April 28, 2007)

Problem 1.
et $ABCD$ be a convex quadrilateral with $AB = BC = CD$, $AC \neq BD$ and let E be the intersection point of its diagonals. Prove that $AE = DE$ if and only if $\angle BAD + \angle ADC = 120^0$.

Solution. Let us first denote $\angle BAC = \angle BCA = \alpha$, $\angle CBD = \angle CDB = \beta$.

Part I : Assume $AE = DE$.

By $\triangle EBC$ we have $\angle AEB = \angle DEC = \alpha + \beta$, thus in $\triangle ABE$ we have $\angle ABE = 180^0 - (2\alpha + \beta)$, and in $\triangle CED$ we have $\angle DCE = 180^0 - (\alpha + 2\beta)$. Then by the law of sines in these two triangles we get

$$\frac{AE}{sin(2\alpha + \beta)} = \frac{AB}{sin(\alpha + \beta)} = \frac{CD}{sin(\alpha + \beta)} = \frac{DE}{sin(\alpha + 2\beta)}.$$

So $sin(2\alpha + \beta) = sin(\alpha + 2\beta)$ with $0^0 < 2\alpha + \beta, \alpha + 2\beta < 180^0$. So either $2\alpha + \beta = \alpha + 2\beta$ or $2\alpha + \beta + \alpha + 2\beta = 180^0$.

The relation $2\alpha+\beta = \alpha+2\beta$ gives $\alpha = \beta$, which in turn implies $\angle BAD = \angle CDA$, and then $\triangle BAD = \triangle CDA$, from which $AC = BD$, a contradiction.

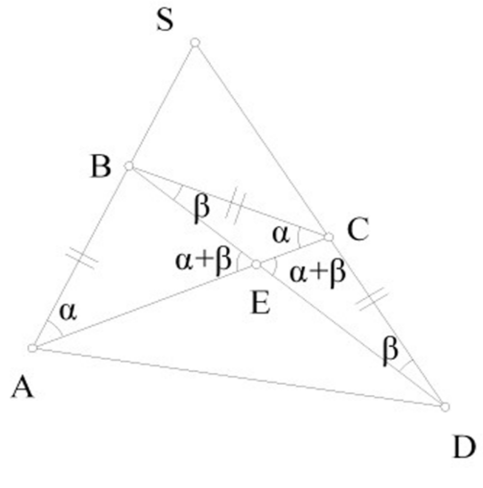
The relation $2\alpha+\beta+\alpha+2\beta = 180^0$ implies $\alpha+\beta = 60^0$. Then $\angle BAD+\angle ADC = \alpha + \angle EAD + \beta + \angle EDA = \alpha + \beta + \angle AEB = 2(\alpha + \beta) = 120^0$.

Part II : Assume $\angle BAD + \angle ADC = 120^0$.

Let S be the intersection point of the lines AB and DC.

As in part I, we have $\angle AEB = \alpha+\beta$. But also $\angle AEB = \angle EAD+\angle EDA$. Thus $2\angle AEB = \alpha + \beta + \angle EAD + \angle EDA = \angle BAD + \angle ADC = 120^0$. i.e $\angle AEB = 60^0$. But $\angle S$ is also 60^0. So $SBEC$ is cyclic. Thus $\angle BSE = \angle BCA = \alpha = \angle SAC$. So $EA = ES$. Similarly $ED = ES$, and the desired result follows. ■

Remark. We can avoid the use of trigonometry in part I, as follows: The triangles BAE and CDE have two pairs of equal sides and their angles AEB, CEC opposite to the sides of one of these pairs also equal. By a well known theorem on the congruence of two triangles, we know then that the angles ABE and DCA opposite to the sides of the other pair are either equal or they add up to 180^0, etc.



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Problem 2.
Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(f(x) - y) + 4f(x)y \quad \text{for any } x, y \in \mathbb{R}.$$

Solution. It is clear that the function $f \equiv 0$ satisfies the given condition.

Assume that $f \not\equiv 0$. Choose x_0 such that $f(x_0) \neq 0$ and set $\widetilde{y} = \dfrac{y}{4f(x_0)}$ for any y. Then plugging in to the given relation x_0 for x and $\widetilde{y}$ for y we get

$$y = f(f(x_0) + \widetilde{y}) - f(f(x_0) - \widetilde{y}). \tag{1}$$

For any y_1, y_2, plugging in to (1) $\dfrac{y_1 - y_2}{2}$ for y, we get

$$\frac{y_1 - y_2}{2} = f(f(x_0) + \widetilde{\frac{y_1 - y_2}{2}}) - f(f(x_0) - \widetilde{\frac{y_1 - y_2}{2}}).$$

In other words, for any y_1, y_2 there exist $x_1(y_1, y_2) = f(x_0) + \widetilde{\frac{y_1-y_2}{2}}$, $x_2(y_1, y_2) = f(x_0) + \widetilde{\frac{y_1-y_2}{2}}$ such that $\dfrac{y_1 - y_2}{2} = f(x_1) - f(x_2)$, i.e. such that

$$2f(x_1) - y_1 = 2f(x_2) - y_2. \tag{2}$$

On the other hand, replacing y by $f(x) - y$ in the given condition gives

$$f(2f(x) - y) = f(y) + 4f(x)(f(x) - y), \text{ i.e.}$$

$$f(y) - y^2 = f(2f(x) - y) - (2(f(x) - y)^2. \tag{3}$$

Now if for any two y_1, y_2, we plug in to (3), $x_1(y_1, y_2)$ and $x_2(y_1, y_2)$ respectively, we get

$$f(y_1) - y_1^2 = f(2f(x_1) - y_1) - (2(f(x_1) - y_1)^2$$

and

$$f(y_2) - y_2^2 = f(2f(x_2) - y_2) - (2(f(x_2) - y_2)^2$$

Then by (2) we get $f(y_1) - y_1^2 = f(y_2) - y_2^2$. Since this happens for any two y_1, y_2, we conclude that $f(x) - x^2 =$ constant for all x, thus $f(x) = x^2 + c, c \in \mathbb{R}$. It is easy to check that such a function satisfies the condition of the problem. ■

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Problem 3. Find all positive integers n such that there is a permutation σ of the set $\{1, 2, ..., n\}$, for which $\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{... + \sqrt{\sigma(n)}}}}$ is a rational number.

Note: A permutation of the set $\{1, 2, ..., n\}$ is a one-to-one function of this set to itself.

Solution. For some $n \in N$, let $\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{... + \sqrt{\sigma(n)}}}} = r_1 \in \mathbb{Q}$. Squaring both sides of the equation we get that $\sqrt{\sigma(2) + \sqrt{\sigma(3) + \sqrt{... + \sqrt{\sigma(n)}}}}$ is also rational. Using the same reasoning recursively, we get that for every $k \in \{1, \ldots, n\}$, $\sqrt{\sigma(k) + \sqrt{\sigma(k+1) + \sqrt{... + \sqrt{\sigma(n)}}}}$ is rational as well. Knowing that the square root of a positive integer is either integer or irrational, we have that $\sqrt{\sigma(n)}$ is integer. Similarly, we get that $\sqrt{\sigma(k) + \sqrt{\sigma(k+1) + \sqrt{... + \sqrt{\sigma(n)}}}}$, for every $k \in \{1, \ldots, n\}$, is integer. Note that for $k = 1$ we get $r_1 \in \mathbb{N}$.

We define a_k as $a_k = \underbrace{\sqrt{n + \sqrt{n + \sqrt{... + \sqrt{n}}}}}_{k}$, for all $k \geq 1$. It is easy to prove by induction that $a_k < \sqrt{n} + 1$, for every $k \geq 1$. Therefore, we have $\sqrt{\sigma(1) + \sqrt{\sigma(2) + \sqrt{... + \sqrt{\sigma(n)}}}} < a_n < \sqrt{n} + 1$, implying $r_1 < \sqrt{n} + 1$.

Let ℓ be the positive integer that satisfies $\ell^2 \leq n < (\ell+1)^2$. For some i, $1 \leq i \leq n$, we have $\sigma(i) = \ell^2$. We distinguish two cases:

First case: $i \neq n$.

Then we have $\ell < \sqrt{\ell^2 + \sqrt{\sigma(i+1) + \sqrt{... + \sqrt{\sigma(n)}}}} < \sqrt{n} + 1 < \ell + 2$, implying

$\sqrt{\ell^2 + \sqrt{\sigma(i+1) + \sqrt{... + \sqrt{\sigma(n)}}}} = \ell + 1$. But then it follows that

$$2\ell + 1 = \sqrt{\sigma(i+1) + \sqrt{... + \sqrt{\sigma(n)}}} < \sqrt{n} + 1 < \ell + 2,$$

giving $\ell < 1$. A contradiction.

<u>*Second case:*</u> $i = n$.

For $\ell > 1$, $\ell^2 - 1$ belongs to the set $\{\sigma(1), ..., \sigma(n-1)\}$. Let $j < n$ be such that $\sigma(j) = \ell^2 - 1$. Similarly to the first case, we have

$$\ell < \sqrt{\ell^2 - 1 + \sqrt{\sigma(j+1) + \sqrt{... + \sqrt{\ell^2}}}} < \sqrt{n} + 1 < \ell + 2,$$

implying $\sqrt{\ell^2 - 1 + \sqrt{\sigma(j+1) + \sqrt{... + \sqrt{\ell^2}}}} = \ell + 1$, and

$$2\ell + 2 = \sqrt{\sigma(j+1) + \sqrt{... + \sqrt{\ell^2}}} < \sqrt{n} + 1 < \ell + 2,$$

a contradiction.

If $\ell = 1$, then $n \in \{1, 2, 3\}$. Checking through all the possibilities, it is easy to see that for $n = 1$ and $n = 3$ there exist permutations that satisfy the initial condition. Namely, for $n = 1$ we have $\sqrt{1} = 1$, and for $n = 3$, we have $\sqrt{2 + \sqrt{3 + \sqrt{1}}} = 2$. For $n = 2$ there is no such permutation.

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Problem 4. For a given positive integer $n > 2$, let C_1, C_2, C_3 be the boundaries of three convex n-gons in the plane such that the sets $C_1 \cap C_2$, $C_2 \cap C_3$, $C_3 \cap C_1$ are finite. Find the maximum number of points of the set $C_1 \cap C_2 \cap C_3$.

Solution 1: Let us first observe that, if a line intersects a convex $n-$gon at finitely many points, then the number of such points is at most 2. Therefore any two of the $n-$gons may intersect in at most $2n$ points. Choose two of the $n-$gons, C_1, C_2, and say that their intersection points are $p_1, p_2, \ldots, p_k$. Thus $k \leq 2n$. Say that the union of the set of vertices of C_1 and C_2 is $\{q_1, q_2, \ldots, q_{2n}\}$. We note that it is possible to have $q_i = q_j$ for some $i \neq j$.

We will define a one-to-one function f from $\{p_1, p_2, \ldots, p_k\}$ to $\{q_1, q_2, \ldots, q_{2n}\}$ as follows. First of all, orient all $n-$gons in the clockwise direction. Thus, if one traverses an $n-$gon according to this orientation, the interior is on the right and the exterior is on the left. For every p_i, there exist precisely two line segments (of non-zero length) which are subsets of C_1 or C_2, say $[q_j, p_i]$ on C_1 and $[q_k, p_i]$ on C_2, such that one can approach to p_i via these line segments in the clockwise direction. Suppose, that the two vectors $p_i - q_j$ and $p_i - q_k$, in this order, form a right handed coordinate system. Then none of the points on $[q_j, p_i]$ can be on or in the interior of C_2, since for any point q on or in the interior of C_2, the vectors $p_i - q$ and $p_i - q_k$ are either positive multiples of each other, or form a left handed coordinate system. In this case we set $f(p_i) = q_j$. Otherwise we set $f(p_i) = q_k$. In both cases, the argument above shows that there are no other intersection points between $f(p_i)$ and p_i, in the clockwise direction. Let us now show that f is $1-1$. If $f(p_i) = f(p_l) = q$ and q say (without loss of generality) belong to C_1, then the first intersection point encountered when one starts from q and traverses C_1 in the clockwise direction has to be both p_i and p_l, hence $p_i = p_l$.

Now let us estimate the number of p_i's that can be contained by the third polygon C_3. Each edge of C_3 contains exactly $0, 1$ or 2 of the p_i's. Suppose that a given edge of C_3 contains 2 of the p_i's, say p_1 and p_2. Since C_1 and C_2 are convex and their intersection with C_3 is generic, they should have vertices between (in the clockwise sense) p_1 and p_2 (with this order), and outside C_3. We claim that at least one of these vertices is not in the set $f(C_1 \cap C_2 \cap C_3)$. Let $q_1 \in C_1$ and $q_2 \in C_2$ respectively be between (in the clockwise sense) p_1 and p_2 (with this order). If q_1 is on or in the interior of C_2 (or q_2 is on or in the interior of C_1), then q_1 (or q_2) is not in the image of f, since recall that $f(p)$ for any $p \in C_1 \cap C_2$ (thus for any $p \in C_1 \cap C_2 \cap C_3$) is a point of one of C_1, C_2 not on or in the interior of the other. So the claim is established in this case. The remaining case is to assume that none of the vertices of any of C_1, C_2 that lie outside C_3 (and between (in the clockwise sense) p_1 and p_2

(with this order)) also lies in the interior of the other of C_1, C_2. In this case clearly the polygons C_1 and C_2 must meet at some point between (in the clockwise sense) p_1 and p_2 (with this order). Say p_3 is the closest to p_1 such point. Then clearly $f(p_3)$ is a vertex of one of C_1, C_2 between (in the clockwise sense) p_1 and p_2 (with this order). These parts of C_1, C_2 though, lie outside C_3; the interior of C_3 lie on the other side of the line p_1p_2. Thus $f(p_3)$ is not in $f(C_1 \cap C_2 \cap C_3)$ and the claim is established in all cases. For the side a of C_3 containing p_1, p_2 let us call $q(a)$ a vertex as the one in the claim we just proved. It is easy to see that for distinct sides a, b of C_3 that contain two of the p's, the points q_a, q_b are distinct. Indeed, let a contain p_1, p_2 and b contain p_3, p_4 among the p's. If one of q_a, q_b belongs to one of C_1, C_2 and the other does not belong to it, we are okay. If both q_a, q_b belong to say C_1, then in a clockwise tour around C_1 starting at p_1, we meet p_1, p_2, p_3, p_4 in this order. If not, say the order is p_1, p_3, p_2, p_4. Then the segments p_1p_2, p_3, p_4 intersect at an interior point, since C_1 is a convex polygon. But then the sides a, b of C_3 have a common interior point, a contradiction. So the correct order is p_1, p_2, p_3, p_4. But we know that adding $q(a), q(b)$ in this tour the correct order is $p_1, q(a), p_2, p_3, q(b), p_4$. Thus $q(a), q(b)$ are distinct as claimed.

Now if x of the edges of C_3 contain 1 of the p_i's and y of them contain 2 of the p_i's, then $x + y \leq$ number of sides of C_3, i.e. $x + y \leq n$. The number of points in $C_1 \cap C_2 \cap C_3$ is $x + 2y$. Since f is injective, $x + 2y$ is also the number of q's in $f(C_1 \cap C_2 \cap C_3)$. Also, by the argument in the previous paragraph, we see that for every distinct edge of C_3 containing 2 points we can assign a corresponding distinct q_i outside the image of $f(C_1 \cap C_2 \cap C_3)$. Therefore x is less or equal to the number of q's that do not belong in $f(C_1 \cap C_2 \cap C_3)$. So $(x + 2y) + y$ is at most as much as the number of q's. I.e. $x + 3y \leq 2n$. Adding this with $x + y \leq n$ and dividing by 2, and also taking into account that $x + 2y$ is an integer

$$x + 2y \leq \lfloor \frac{3n}{2} \rfloor$$

Let us now show that this is the best upper bound for every $n \geq 3$. One way (among many) to construct an example is as follows: Construct two regular $n-$gons C_1, C_2 with the same center, such that their intersection points form a regular $2n-$gon. Call the vertices $p_1, p_2, \ldots, p_{2n}$ in a cyclic order. Let the circumcircle of this $2n-$gon be $\mathcal{C}$. Then let the $n-$gon bounded by the lines $p_1p_3, p_5p_7, p_9p_{11}, \ldots$ (including $p_{2k+1}p_1$ in case n is an odd $n = 2k + 1$) together with the tangent lines to $\mathcal{C}$ at $p_4, p_8, p_{12}, \ldots$ be C_3. It can easily be checked that $|C_1 \cap C_2 \cap C_3| = \lfloor \frac{3n}{2} \rfloor$. ■

Solution 2: Let A and B be two consequtive points of $C_1 \cap C_2 \cap C_3$ observed in the clockwise direction from a point in the interior of all three n-gons. Let's look for each C_i its section in the clockwise direction between A and B exluding these points. If some two of these sections both do not contain any vertices of their corresponding n-gons, then the segment AB belongs to both n-gons, a contradiction.

Thus at least two of these segments have at least one vertex each, and moreoverthey do not contain the segment. Trivially, two distinct such vertices exist. Since there exist $|C_1 \cap C_2 \cap C_3|$ many consequtive points points A and B of $C_1 \cap C_2 \cap C_3$, there should exist at least $2|C_1 \cap C_2 \cap C_3|$ distinct vertices of the three n-gons. Thus $2|C_1 \cap C_2 \cap C_3| \leq 3n$ i.e. $|C_1 \cap C_2 \cap C_3| \leq \lfloor \frac{3n}{2} \rfloor$ since ($|C_1 \cap C_2 \cap C_3|$ is an integer as well).

Actually we can achieve this upper bound by the example given in the Solution 1.