Trigonometrija trougla 1

Notacija:

-Neka je $\triangle ABC$ pravougli trougao u kojem je $\angle C = 90°$.Tada je $\sin\angle A = \frac{a}{c}$ i $\cos\angle A = \frac{b}{c}$

-$\tan\angle A = \frac{\sin\angle A}{\cos\angle A}, \cot\angle A = \frac{\cos\angle A}{\sin\angle A}$ su funkcije tangensa i kotangensa

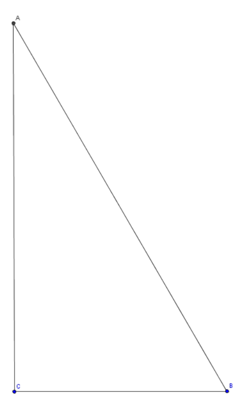
Stavovi trigonometrije trougla:

- $\sin^2\alpha + \cos^2\alpha = 1$
- $\sin(\pi - \alpha) = \sin\alpha,\ \cos(\pi - \alpha) = -\cos\alpha$
- $\sin(\frac{\pi}{2} - \alpha) = \cos\alpha,\ \sin(\frac{\pi}{2} + \alpha) = \cos\alpha$
- $\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$
- $\sin\alpha \cdot \sin\beta = \frac{1}{2} \cdot \{\cos(\alpha - \beta) - \cos(\alpha + \beta)\}$
- $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2} \cdot \cos\frac{\alpha-\beta}{2}$
- $\sin 0° = 0,\ \sin 90° = 1$

-U trouglu $\triangle ABC$ je $\frac{a}{\sin\angle A} = \frac{b}{\sin\angle B} = \frac{c}{\sin\angle C} = 2R$
-U trouglu $\triangle ABC$ je $c^2 = a^2 + b^2 - 2ab \cdot \cos\angle C$

Primjer 1. Pokazati da je $\sin 30° = \frac{1}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$

Rjesenje:



Neka je $\triangle ABC$ pravougli trougao i $\angle C = 90°, \angle A = 30°, \angle B = 60°$.Ocito je $\triangle ABC$ pola jednakostranicnog trougla pa je

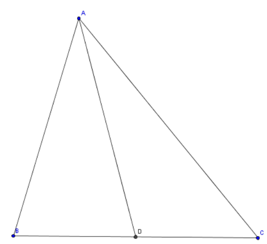
$$\sin 30° = \frac{a}{c} = \frac{1}{2}$$

Koristeci Pitagorinu teoremu je

$$\sin 60^\circ = \frac{b}{c} = \frac{\sqrt{c^2 - a^2}}{c} = \frac{\sqrt{\frac{3c^2}{4}}}{c} = \frac{\sqrt{3}}{2}$$

Primjer 2. Pokazati da je $m_a = \sqrt{\frac{2b^2 + 2c^2 - a^2}{4}}$

Rjesenje:



Neka je D sredina stranice BC .Iz kosinusnih teorema na trouglove $\triangle ABC$ i $\triangle ABD$ imamo

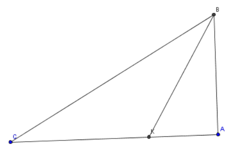
$$m_a^2 = c^2 + (\frac{a}{2})^2 - ac \cdot \cos\beta = c^2 + \frac{a^2}{4} - ac \cdot \frac{a^2 + c^2 - b^2}{2ac} =$$

$$= c^2 + \frac{a^2}{4} - \frac{a^2 + c^2 - b^2}{2} = \frac{2b^2 + 2c^2 - b^2}{4} \Rightarrow$$

$$m_a = \sqrt{\frac{2b^2 + 2c^2 - a^2}{4}}$$

Primjer 3. Neka je $\triangle ABC$ dani trougao i BK simetrala iz vrha B ,i neka je $AK = 1, BK = KC = 2$.Odrediti uglove trougla.

Rjesenje:



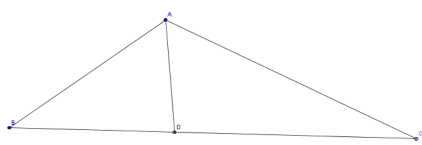
Neka je $\angle KBC = \alpha$.Tada je $\angle KCB = \angle KBA = \alpha$pa je $\angle BAC = 180^\circ - 3\alpha$ pa imamo

$$\frac{1}{\sin\alpha} = \frac{2}{\sin(3\alpha)} \Rightarrow$$

$$3\sin\alpha - 4\sin^3\alpha = 2\sin\alpha \Rightarrow \sin^2\alpha = \frac{1}{4} \Rightarrow \alpha = 30^\circ$$

Primjer 4. Ako je $\frac{1}{b} + \frac{1}{c} = \frac{1}{\omega_a}$ pokazati da je $\angle A = 120^\circ$.

Rjesenje:



Neka je $\angle A = \alpha, \angle B = \beta, \angle C = \gamma$ i neka je D podnozje simetrale iz ugla $\angle A$ na stranici BC.Ii sinusnih teorema na $\triangle ADC$ i $\triangle BDC$ je

$$\frac{\omega_a}{\sin\beta} = \frac{c}{\sin(\pi - \frac{\alpha}{2} - \beta)} \Rightarrow \frac{1}{c} = \frac{\sin\beta}{\omega_a \cdot \sin(\frac{\alpha}{2} + \beta)}$$

$$\frac{\omega_a}{\sin\gamma} = \frac{b}{\sin(\pi - \frac{\alpha}{2} - \gamma)} \Rightarrow \frac{1}{b} = \frac{\sin\gamma}{\omega_a \cdot \sin(\frac{\alpha}{2} + \gamma)}$$

Sabirajuci ove dvije jednakosti je

$$\omega_a(\frac{1}{b}+\frac{1}{c})=\frac{\sin\beta}{\sin(\frac{\alpha}{2}+\beta)}+\frac{\sin\gamma}{\sin(\frac{\alpha}{2}+\gamma)}=$$

$$=\frac{\sin\beta\cdot\sin(\frac{\alpha}{2}+\gamma)+\sin\gamma\cdot\sin(\frac{\alpha}{2}+\beta)}{\sin(\frac{\alpha}{2}+\beta)\cdot\sin(\frac{\alpha}{2}+\gamma)}=\frac{\frac{1}{2}\cdot\{\cos(\frac{\alpha}{2}+\gamma-\beta)-\cos(\frac{\alpha}{2}+\gamma+\beta)+\cos(\frac{\alpha}{2}+\beta-\gamma)-\cos(\frac{\alpha}{2}+\beta+\gamma)\}}{\frac{1}{2}\cdot\{\cos(\beta-\gamma)-\cos(\alpha+\beta+\gamma)\}}$$

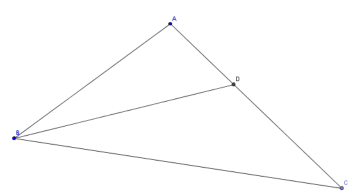
$$=\frac{\cos(\frac{\alpha}{2}+\gamma-\beta)+\cos(\frac{\alpha}{2}+\beta-\gamma)+2\cdot\cos\frac{\alpha}{2}}{\cos(\beta-\gamma)+1}=\frac{2\cdot\cos\frac{\alpha}{2}\cdot\cos(\beta-\gamma)+2\cdot\cos\frac{\alpha}{2}}{\cos(\beta-\gamma)+1}=2\cdot\cos\frac{\alpha}{2}$$

Koristeci uslov je sada

$$2\cos\frac{\alpha}{2}=1\Rightarrow\alpha=120^\circ$$

Primjer 5. Neka je $\triangle ABC$ jednakokraki trougao i $AB=AC$.Neka simetrala ugla $\angle B$ sijece AC u D.Ako je $BC=BD+AD$ odrediti ugao $\angle A$

Rjesenje:



Neka je $\angle ABD=\angle DBC=\alpha$

$$BC=\frac{BD\cdot\sin(3\alpha)}{\sin(2\alpha)}, AD=\frac{BD\cdot\sin\alpha}{\sin(4\alpha)}$$

$$BC=BD+AD\Leftrightarrow$$

$$\frac{BD\cdot\sin(3\alpha)}{\sin(2\alpha)}=BD+\frac{BD\cdot\sin\alpha}{\sin(4\alpha)}\Leftrightarrow$$

$$\frac{\sin(3\alpha)}{\sin(2\alpha)}=1+\frac{\sin\alpha}{\sin(4\alpha)}$$

$$2\cos(2\alpha)\cdot\sin(3\alpha)=\sin(4\alpha)+\sin(\alpha)\Leftrightarrow$$

$$\sin(5\alpha) + \sin\alpha = \sin(4\alpha) + \sin\alpha \Leftrightarrow$$

$$\sin(5\alpha) = \sin(4\alpha) \Rightarrow$$

$$5\alpha + 4\alpha = 180^\circ \Rightarrow \alpha = 20^\circ \Rightarrow \angle A = 100^\circ$$

-

Zadaci za samostalan rad:

1.Pokazati da je $\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}$,odrediti $\sin 15^\circ$ i $\sin 75^\circ$.

2.Neka je ω_a duzina simetrale iz vrha A.Pokazati da je $\omega_a = \sqrt{\frac{(a+b+c)(b+c-a)bc}{(b+c)^2}}$
.

3. Neka je $\triangle ABC$ dani trougao i $\angle BAC = 120^\circ, BC = 2AC$ .Odrediti ugao izmedju tezisnica AD i BE.

4. Neka je $\triangle ABC$ dani trougao i $\angle A = 30^\circ, \angle B = 80^\circ$.Neka je M tacka u trouglu takva da je $\angle MAC = 10^\circ, \angle MCA = 30^\circ$.Odrediti $\angle BMC$.

5. Neka je $\triangle ABC$ jednakokraki trougao $AB = AC$ i $\angle A = 20^\circ$.Neka je D tacka na AC takva da je $\angle DBC = 25^\circ$ i neka je E tacka na AB takva da je $\angle BCE = 65^\circ$ .Odrediti ugao $\angle CED$.