USA and International Mathematical Olympiads 2006-2007

Edited by

Zuming Feng   Yufei Zhao

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1. Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p}$$

if and only if s is not a divisor of $p-1$.

(For x a real number, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x.)

First Solution: First suppose that s is a divisor of $p-1$; write $d = (p-1)/s$. As x varies among $1, 2, \ldots, p-1$, $\{sx/p\}$ takes the values $1/p, 2/p, \ldots, (p-1)/p$ once each in some order. The possible values with $\{sx/p\} < s/p$ are precisely $1/p, \ldots, (s-1)/p$. From the fact that $\{sd/p\} = (p-1)/p$, we realize that the values $\{sx/p\} = (p-1)/p, (p-2)/p, \ldots, (p-s+1)/p$ occur for

$$x = d, 2d, \ldots, (s-1)d$$

(which are all between 0 and p), and so the values $\{sx/p\} = 1/p, 2/p, \ldots, (s-1)/p$ occur for

$$x = p-d, p-2d, \ldots, p-(s-1)d,$$

respectively. From this it is clear that m and n cannot exist as requested.

Conversely, suppose that s is not a divisor of $p-1$. Put $m = \lceil p/s \rceil$; then m is the smallest positive integer such that $\{ms/p\} < s/p$, and in fact $\{ms/p\} = (ms-p)/p$. However, we cannot have $\{ms/p\} = (s-1)/p$ or else $(m-1)s = p-1$, contradicting our hypothesis that s does not divide $p-1$. Hence the unique $n \in \{1, \ldots, p-1\}$ for which $\{nx/p\} = (s-1)/p$ has the desired properties (since the fact that $\{nx/p\} < s/p$ forces $n \geq m$, but $m \neq n$).

Second Solution: We prove the contrapositive statement:

> Let p be a prime number and let s be an integer with $0 < s < p$. Prove that the following statements are equivalent:
>
> (a) s is a divisor of $p-1$;
>
> (b) if integers m and n are such that $0 < m < p$, $0 < n < p$, and
>
> $$\left\{\frac{sm}{p}\right\} < \left\{\frac{sn}{p}\right\} < \frac{s}{p},$$
>
> then $0 < n < m < p$.

Since p is prime and $0 < s < p$, s is relatively prime to p and

$$S = \{s, 2s, \ldots, (p-1)s, ps\}$$

is a set of complete residues classes modulo p. In particular,

(1) there is an unique integer d with $0 < d < p$ such that $sd \equiv -1 \pmod{p}$; and

(2) for every k with $0 < k < p$, there exists a unique pair of integers (m_k, a_k) with $0 < m_k < p$ such that $m_k s + a_k p = k$.

Now we consider the equations

$$m_1 s + a_1 p = 1,\ m_2 s + a_2 p = 2,\ \dots\ , m_s s + a_s p = s.$$

Hence $\{m_k s/p\} = k/p$ for $1 \le k \le s$.

Statement (b) holds if and only $0 < m_s < m_{s-1} < \cdots < m_1 < p$. For $1 \le k \le s-1$, $m_k s - m_{k+1} s = (a_{k+1} - a_k)p - 1$, or $(m_k - m_{k+1})s \equiv -1 \pmod p$. Since $0 < m_{k+1} < m_k < p$, by (1), we have $m_k - m_{k+1} = d$. We conclude that (b) holds if and only if $m_s, m_{s-1}, \dots, m_1$ form an arithmetic progression with common difference $-d$. Clearly $m_s = 1$, so $m_1 = 1 + (s-1)d = jp - d + 1$ for some j. Then $j = 1$ because m_1 and d are both positive and less than p, so $sd = p - 1$. This proves (a).

Conversely, if (a) holds, then $sd = p - 1$ and $m_k \equiv -dsm_k \equiv -dk \pmod p$. Hence $m_k = p - dk$ for $1 \le k \le s$. Thus $m_s, m_{s-1}, \dots, m_1$ form an arithmetic progression with common difference $-d$. Hence (b) holds.

(This problem was proposed by Kiran Kedlaya.)

2. For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k+1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $N/2$.

Solution: The minimum is $N = 2k^3 + 3k^2 + 3k$. The set

$$\{k^2+1, k^2+2, \dots, k^2+2k+1\}$$

has sum $2k^3 + 3k^2 + 3k + 1 = N + 1$ which exceeds N, but the sum of the k largest elements is only $(2k^3 + 3k^2 + 3k)/2 = N/2$. Thus this N is such a value.

Suppose $N < 2k^3 + 3k^2 + 3k$ and there are positive integers $a_1 < a_2 < \cdots < a_{2k+1}$ with $a_1 + a_2 + \cdots + a_{2k+1} > N$ and $a_{k+2} + \cdots + a_{2k+1} \le N/2$. Then

$$(a_{k+1}+1) + (a_{k+1}+2) + \cdots + (a_{k+1}+k) \le a_{k+2} + \cdots + a_{2k+1} \le N/2 < \frac{2k^3+3k^2+3k}{2}.$$

This rearranges to give $2ka_{k+1} \le N - k^2 - k$ and $a_{k+1} < k^2 + k + 1$. Hence $a_{k+1} \le k^2 + k$. Combining these we get

$$2(k+1)a_{k+1} \le N + k^2 + k.$$

We also have

$$(a_{k+1} - k) + \cdots + (a_{k+1} - 1) + a_{k+1} \ge a_1 + \cdots + a_{k+1} > N/2$$

or $2(k+1)a_{k+1} > N + k^2 + k$. This contradicts the previous inequality, hence no such set exists for $N < 2k^3 + 3k^2 + 3k$ and the stated value is the minimum.

(This problem was proposed by Dick Gibbs.)

3. For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence $\{p(f(n^2)) - 2n\}_{n \ge 0}$ is bounded above. (In particular, this requires $f(n^2) \ne 0$ for $n \ge 0$.)

Solution: The polynomial f has the required properties if and only if

$$f(x) = c(4x - a_1^2)(4x - a_2^2)\cdots(4x - a_k^2), \qquad (*)$$

where $a_1, a_2, \ldots, a_k$ are odd positive integers and c is a nonzero integer. It is straightforward to verify that polynomials given by $(*)$ have the required property. If p is a prime divisor of $f(n^2)$ but not of c, then $p|(2n - a_j)$ or $p|(2n + a_j)$ for some $j \le k$. Hence $p - 2n \le \max\{a_1, a_2, \ldots, a_k\}$. The prime divisors of c form a finite set and do affect whether or not the given sequence is bounded above. The rest of the proof is devoted to showing that any f for which $\{p(f(n^2)) - 2n\}_{n\ge 0}$ is bounded above is given by $(*)$.

Let $\mathbb{Z}[x]$ denote the set of all polynomials with integral coefficients. Given $f \in \mathbb{Z}[x]$, let $\mathcal{P}(f)$ denote the set of those primes that divide at least one of the numbers in the sequence $\{f(n)\}_{n\ge 0}$. The solution is based on the following lemma.

Lemma *If $f \in \mathbb{Z}[x]$ is a nonconstant polynomial then $\mathcal{P}(f)$ is infinite.*

Proof: Repeated use will be made of the following basic fact: if a and b are distinct integers and $f \in \mathbb{Z}[x]$, then $a - b$ divides $f(a) - f(b)$. If $f(0) = 0$, then p divides $f(p)$ for every prime p, so $\mathcal{P}(f)$ is infinite. If $f(0) = 1$, then every prime divisor p of $f(n!)$ satisfies $p > n$. Otherwise p divides $n!$, which in turn divides $f(n!) - f(0) = f(n!) - 1$. This yields $p|1$, which is false. Hence $f(0) = 1$ implies that $\mathcal{P}(f)$ is infinite. To complete the proof, set $g(x) = f(f(0)x)/f(0)$ and observe that $g \in \mathbb{Z}[x]$ and $g(0) = 1$. The preceding argument shows that $\mathcal{P}(g)$ is infinite, and it follows that $\mathcal{P}(f)$ is infinite. ∎

Suppose $f \in \mathbb{Z}[x]$ is nonconstant and there exists a number M such that $p(f(n^2)) - 2n \le M$ for all $n \ge 0$. Application of the lemma to $f(x^2)$ shows that there is an infinite sequence of distinct primes $\{p_j\}$ and a corresponding infinite sequence of nonnegative integers $\{k_j\}$ such that $p_j|f(k_j^2)$ for all $j \ge 1$. Consider the sequence $\{r_j\}$ where $r_j = \min\{k_j \pmod{p_j}, p_j - k_j \pmod{p_j}\}$. Then $0 \le r_j \le (p_j - 1)/2$ and $p_j|f(r_j^2)$. Hence $2r_j + 1 \le p_j \le p(f(r_j^2)) \le M + 2r_j$, so $1 \le p_j - 2r_j \le M$ for all $j \ge 1$. It follows that there is an integer a_1 such that $1 \le a_1 \le M$ and $a_1 = p_j - 2r_j$ for infinitely many j. Let $m = \deg f$. Then $p_j|4^m f((p_j - a_1)/2)^2)$ and $4^m f((x - a_1)/2)^2) \in \mathbb{Z}[x]$. Consequently, $p_j|f((a_1/2)^2)$ for infinitely many j, which shows that $(a_1/2)^2$ is a zero of f. Since $f(n^2) \ne 0$ for $n \ge 0$, a_1 must be odd. Then $f(x) = (4x - a_1^2)g(x)$ where $g \in \mathbb{Z}[x]$. (See the note below.) Observe that $\{p(g(n^2)) - 2n\}_{n\ge 0}$ must be bounded above. If g is constant, we are done. If g is nonconstant, the argument can be repeated to show that f is given by $(*)$.

Note: The step that gives $f(x) = (4x - a_1^2)g(x)$ where $g \in \mathbb{Z}[x]$ follows immediately using a lemma of Gauss. The use of such an advanced result can be avoided by first writing $f(x) = r(4x - a_1^2)g(x)$ where r is rational and $g \in \mathbb{Z}[x]$. Then continuation gives $f(x) = c(4x - a_1^2) \cdots (4x - a_k^2)$ where c is rational and the a_i are odd. Consideration of the leading coefficient shows that the denominator of c is 2^s for some $s \ge 0$ and consideration of the constant term shows that the denominator is odd. Hence c is an integer.

(This problem was proposed by Titu Andreescu and Gabriel Dospinescu.)

4. Find all positive integers n such that there are $k \ge 2$ positive rational numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 + a_2 + \cdots + a_k = a_1 \cdot a_2 \cdots a_k = n$.

Solution: The answer is $n = 4$ or $n \ge 6$.

I. First, we prove that each $n \in \{4, 6, 7, 8, 9, \ldots\}$ *satisfies* the condition.

(1). If $n = 2k \geq 4$ is *even*, we set $(a_1,\ a_2,\ \ldots,\ a_k) = (k,\ 2,\ 1,\ \ldots,\ 1)$:

$$a_1 + a_2 + \ \ldots \ + a_k \ = \ k + 2 + 1 \cdot (k-2) \ = \ 2k = n,$$

and

$$a_1 \cdot a_2 \cdot \ \ldots \ \cdot a_k = 2k = n \ .$$

(2). If $n = 2k + 3 \geq 9$ is *odd*, we set $(a_1,\ a_2,\ \ldots,\ a_k) = \left(k + \frac{3}{2},\ \frac{1}{2},\ 4,\ 1,\ \ldots,\ 1\right)$:

$$a_1 + a_2 + \ \ldots \ + a_k \ = \ k + \frac{3}{2} + \frac{1}{2} + 4 + (k-3) \ = \ 2k + 3 = n,$$

and

$$a_1 \cdot a_2 \cdot \ \ldots \ \cdot a_k = \left(k + \frac{3}{2}\right) \cdot \frac{1}{2} \cdot 4 = 2k + 3 = n \ .$$

(3). A very special case is $n = 7$, in which we set $(a_1,\ a_2,\ a_3) = \left(\frac{4}{3},\ \frac{7}{6},\ \frac{9}{2}\right)$. It is also easy to check that

$$a_1 + a_2 + a_3 = a_1 \cdot a_2 \cdot a_3 = 7 = n.$$

II. Second, we prove by contradiction that each $n \in \{1, 2, 3, 5\}$ *fails to satisfy* the condition.

Suppose, on the contrary, that there is a set of $k \geq 2$ positive rational numbers whose sum and product are both $n \in \{1, 2, 3, 5\}$. By the Arithmetic-Geometric Mean inequality, we have

$$n^{1/k} = \sqrt[k]{a_1 \cdot a_2 \cdot \ \ldots \ \cdot a_k} \leq \frac{a_1 + a_2 + \ \ldots \ + a_k}{k} = \frac{n}{k} \ ,$$

which gives

$$n \geq k^{\frac{k}{k-1}} = k^{1 + \frac{1}{k-1}} \ .$$

Note that $n > 5$ whenever $k = 3,\ 4$, or $k \geq 5$:

$$k = 3 \quad \Rightarrow \quad n \geq 3\sqrt{3} = 5.196... > 5;$$

$$k = 4 \quad \Rightarrow \quad n \geq 4\sqrt[3]{4} = 6.349... > 5;$$

$$k \geq 5 \quad \Rightarrow \quad n \geq 5^{1 + \frac{1}{k-1}} > 5 \ .$$

This proves that none of the integers 1, 2, 3, or 5 can be represented as the sum and, at the same time, as the product of three or more positive numbers $a_1, a_2, \ldots, a_k$, rational or irrational.

The remaining case $k = 2$ also goes to a contradiction. Indeed, $a_1 + a_2 = a_1 a_2 = n$ implies that $n = a_1^2/(a_1 - 1)$ and thus a_1 satisfies the quadratic

$$a_1^2 - n a_1 + n = 0 \ .$$

Since a_1 is supposed to be *rational*, the discriminant $n^2 - 4n$ must be a perfect square. However, it can be easily checked that this is not the case for any $n \in \{1, 2, 3, 5\}$. This completes the proof.

Note: Actually, among all positive integers only $n = 4$ can be represented both as the sum and product of the same two rational numbers. Indeed, $(n-3)^2 < n^2 - 4n = (n-2)^2 - 4 < (n-2)^2$ whenever $n \geq 5$; and $n^2 - 4n < 0$ for $n = 1, 2, 3$.

(This problem was proposed by Ricky Liu.)

5. A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n+2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

First Solution: For $i \geq 0$ and $k \geq 1$, let $x_{i,k}$ denote the minimum number of jumps needed to reach the integer $n_{i,k} = 2^i k$. We must prove that

$$x_{i,k} > x_{i,1} \qquad (*)$$

for all $i \geq 0$ and $k \geq 2$. We prove this using the method of descent.

First note that $(*)$ holds for $i = 0$ and all $k \geq 2$, because it takes 0 jumps to reach the starting value $n_{0,1} = 1$, and at least one jump to reach $n_{0,k} = k \geq 2$. Now assume that that $(*)$ is not true for all choices of i and k. Let i_0 be the minimal value of i for which $(*)$ fails for some k, let k_0 be the minimal value of $k > 1$ for which $x_{i_0,k} \leq x_{i_0,1}$. Then it must be the case that $i_0 \geq 1$ and $k_0 \geq 2$.

Let J_{i_0,k_0} be a shortest sequence of $x_{i_0,k_0} + 1$ integers that the frog occupies in jumping from 1 to $2^{i_0}k_0$. The length of each jump, that is, the difference between consecutive integers in J_{i_0,k_0}, is either 1 or a positive integer power of 2. The sequence J_{i_0,k_0} cannot contain 2^{i_0} because it takes more jumps to reach $2^{i_0}k_0$ than it does to reach 2^{i_0}. Let 2^{M+1}, $M \geq 0$ be the length of the longest jump made in generating J_{i_0,k_0}. Such a jump can only be made from a number that is divisible by 2^M (and by no higher power of 2). Thus we must have $M < i_0$, since otherwise a number divisible by 2^{i_0} is visited before $2^{i_0}k_0$ is reached, contradicting the definition of k_0.

Let 2^{m+1} be the length of the jump when the frog jumps over 2^{i_0}. If this jump starts at $2^m(2t-1)$ for some positive integer t, then it will end at $2^m(2t-1) + 2^{m+1} = 2^m(2t+1)$. Since it goes over 2^{i_0} we see $2^m(2t-1) < 2^{i_0} < 2^m(2t+1)$ or $(2^{i_0-m}-1)/2 < t < (2^{i_0-m}+1)/2$. Thus $t = 2^{i_0-m-1}$ and the jump over 2^{i_0} is from $2^m(2^{i_0-m}-1) = 2^{i_0} - 2^m$ to $2^m(2^{i_0-m}+1) = 2^{i_0} + 2^m$.

Considering the jumps that generate J_{i_0,k_0}, let N_1 be the number of jumps from 1 to $2^{i_0} + 2^m$, and let N_2 be the number of jumps from $2^{i_0} + 2^m$ to $2^{i_0}k$. By definition of i_0, it follows that 2^m can be reached from 1 in less than N_1 jumps. On the other hand, because $m < i_0$, the number $2^{i_0}(k_0-1)$ can be reached from 2^m in exactly N_2 jumps by using the same jump length sequence as in jumping from $2^m + 2^{i_0}$ to $2^{i_0}k_0 = 2^{i_0}(k_0-1) + 2^i_0$. The key point here is that the shift by 2^{i_0} does not affect any of divisibility conditions needed to make jumps of the same length. In particular, with the exception of the last entry, $2^{i_0}k_0$, all of the elements of J_{i_0,k_0} are of the form $2^p(2t+1)$ with $p < i_0$, again because of the definition of k_0. Because $2^p(2t+1) - 2^{i_0} = 2^p(2t - 2^{i_0-p} + 1)$ and the number $2t + 2^{i_0-p} + 1$ is odd, a jump of size 2^{p+1} can be made from $2^p(2t+1) - 2^{i_0}$ just as it can be made from $2^p(2t+1)$.

Thus the frog can reach 2^m from 1 in less than N_1 jumps, and can then reach $2^{i_0}(k_0-1)$ from 2^m in N_2 jumps. Hence the frog can reach $2^{i_0}(k_0-1)$ from 1 in less than $N_1 + N_2$ jumps, that is, in fewer jumps than needed to get to $2^{i_0}k_0$ and hence in fewer jumps than required to get to 2^{i_0}. This contradicts the definition of k_0.

Second Solution: Suppose $x_0 = 1, x_1, \ldots, x_t = 2^i k$ are the integers visited by the frog on his trip from 1 to $2^i k$, $k \geq 2$. Let $s_j = x_j - x_{j-1}$ be the jump sizes. Define a reduced path y_j inductively by

$$y_j = \begin{cases} y_{j-1} + s_j & \text{if } y_{j-1} + s_j \leq 2^i, \\ y_{j-1} & \text{otherwise.} \end{cases}$$

Say a jump s_j is deleted in the second case. We will show that the distinct integers among the y_j give a shorter path from 1 to 2^i. Clearly $y_j \leq 2^i$ for all j. Suppose $2^i - 2^{r+1} < y_j \leq 2^i - 2^r$ for some $0 \leq r \leq i-1$. Then every deleted jump before y_j must have length greater than 2^r, hence must be a multiple of 2^{r+1}. Thus $y_j \equiv x_j \pmod{2^{r+1}}$. If $y_{j+1} > y_j$, then either $s_{j+1} = 1$ (in which case this is a valid jump) or $s_{j+1}/2 = 2^m$ is the exact power of 2 dividing x_j. In the second case, since $2^r \geq s_{j+1} > 2^m$, the congruence says 2^m is also the exact power of 2 dividing y_j, thus again this is a valid jump. Thus the distinct y_j form a valid path for the frog. If $j = t$ the congruence gives $y_t \equiv x_t \equiv 0 \pmod{2^{r+1}}$, but this is impossible for $2^i - 2^{r+1} < y_t \leq 2^i - 2^r$. Hence we see $y_t = 2^i$, that is, the reduced path ends at 2^i. Finally since the reduced path ends at $2^i < 2^i k$ at least one jump must have been deleted and it is strictly shorter than the original path.

Third Solution: (By Brian Lawrence) Suppose $2^i k$ can be reached in m jumps.

Our approach will be to consider the frog's life as a sequence of leaps of certain lengths. We will prove that by removing the longest leaps from the sequence, we generate a valid sequence of leaps that ends at 2^i. Clearly this sequence will be shorter, since it was obtained by removing leaps. The result will follow.

Lemma *If we remove the longest leap in the frog's life (or one of the longest, in case of a tie) the sequence of leaps will still be legitimate.*

Proof: By definition, a leap from n to $n + \nu$ is legitimate if and only if either (a) $\nu = 1$, or (b) $\nu = 2^{m_n+1}$. If all leaps are of length 1, then clearly removing one leap does not make any others illegitimate; suppose the longest leap has length 2^s.

Then we remove this leap and consider the effect on all the other leaps. Take an arbitrary leap starting (originally) at n, with length ν. Then $\nu \leq 2^s$. If $\nu = 1$ the new leap is legitimate no matter where it starts. Say $\nu > 1$. Then $\nu = 2^{m_n+1}$. Now if the leap is before the removed leap, its position is not changed, so $\nu = 2^{m_n+1}$ and it remains legitimate. If it is after the removed leap, its starting point is moved back to $n - 2^s$. Now since $2^{m_n+1} = \nu \leq 2^s$, we have $m_n \leq s-1$; that is, 2^s does not divide n. Therefore, 2^{m_n} is the highest power of 2 dividing $n - 2^s$, so $\nu = 2^{m_{n-2^s}+1}$ and the leap is still legitimate. This proves the Lemma. ∎

We now remove leaps from the frog's sequence of leaps in decreasing order of length. The frog's path has initial length $2^i k - 1$; we claim that at some point its length is $2^i - 1$.

Let the frog's m leaps have lengths

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_m.$$

Define a function f by

$$f(0) = 2^i k$$

$$f(i) = f(i-1) - a_i, 1 \leq i \leq m.$$

Clearly $f(i)$ is the frog's final position if we remove the i longest leaps. Note that $f(m) = 1$ – if we remove all leaps, the frog ends up at 1. Let $f(j)$ be the last value of f that is at least 2^i. That is, suppose $f(j) \geq 2^i$, $f(j+1) < 2^i$. Now we have $a_{j+1} | a_k$ for all $k \leq j$ since $\{a_k\}$ is a decreasing sequence of powers of 2. If $a_{j+1} > 2^i$, we have $2^i | a_p$ for $p \leq j$, so $2^i | f(j+1)$. But $0 < f(j+1) < 2^i$, contradiction. Thus $a_{j+1} \leq 2^i$, so, since a_{j+1} is a power of two, $a_{j+1} | 2^i$. Since $a_{j+1} | 2^i k$ and $a_1, \cdots, a_j$, we know that $a_{j+1} | f(j)$, and $a_{j+1} | f(j+1)$. So $f(j+1)$, $f(j)$ are two consecutive multiples of a_{j+1}, and 2^i (another such multiple) satisfies $f(j+1) < 2^i \leq f(j)$. Thus we have $2^i = f(j)$, so by removing j leaps we make a path for the frog that is legitimate by the Lemma, and ends on 2^i.

Now let m be the minimum number of leaps needed to reach $2^i k$. Applying the Lemma and the argument above the frog can reach 2^i in only $m - j$ leaps. Since $j > 0$ trivially ($j = 0$ implies $2^i = f(j) = f(0) = 2^i k$) we have $m - j < m$ as desired.

(This problem was proposed by Zoran Sunik.)

6. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

First Solution: Let P be the second intersection of the circumcircles of triangles TCF and TDE. Because the quadrilateral $PEDT$ is cyclic, $\angle PET = \angle PDT$, or

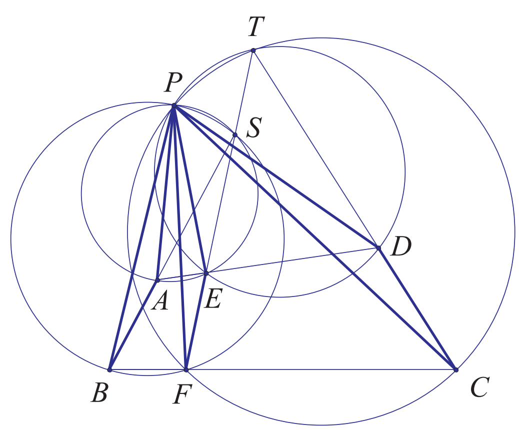
$$\angle PEF = \angle PDC. \qquad (*)$$

Because the quadrilateral $PFCT$ is cyclic,

$$\angle PFE = \angle PFT = \angle PCT = \angle PCD. \qquad (**)$$

By equations $(*)$ and $(**)$, it follows that triangle PEF is similar to triangle PDC. Hence $\angle FPE = \angle CPD$ and $PF/PE = PC/PD$. Note also that $\angle FPC = \angle FPE + \angle EPC = \angle CPD + \angle EPC = \angle EPD$. Thus, triangle EPD is similar to triangle FPC. Another way to say this is that there is a spiral similarity centered at P that sends triangle PFE to triangle PCD, which implies that there is also a spiral similarity, centered at P, that sends triangle PFC to triangle PED, and vice versa. In terms of complex numbers, this amounts to saying that

$$\frac{D-P}{E-P} = \frac{C-P}{F-P} \Longrightarrow \frac{E-P}{F-P} = \frac{D-P}{C-P}.$$



Because $AE/ED = BF/FC$, points A and B are obtained by extending corresponding segments of two similar triangles PED and PFC, namely, DE and CF, by the identical proportion. We conclude that triangle PDA is similar to triangle PCB, implying that triangle PAE is similar to triangle PBF. Therefore, as shown before, we can establish the similarity between triangles PBA and PFE, implying that

$$\angle PBS = \angle PBA = \angle PFE = \angle PFS \quad \text{and} \quad \angle PAB = \angle PEF.$$

The first equation above shows that $PBFS$ is cyclic. The second equation shows that $\angle PAS = 180^\circ - \angle BAP = 180^\circ - \angle FEP = \angle PES$; that is, $PAES$ is cyclic. We conclude that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through point P.

Note. There are two spiral similarities that send segment EF to segment CD. One of them sends E and F to D and C, respectively; the point P is the center of this spiral similarity. The other sends E and F to C and D, respectively; the center of this spiral similarity is the second intersection (other than T) of the circumcircles of triangles TFD and TEC.

Second Solution: We will give a solution using complex coordinates. The first step is the following lemma.

Lemma *Suppose s and t are real numbers and x, y and z are complex. The circle in the complex plane passing through x, $x + ty$ and $x + (s + t)z$ also passes through the point $x + syz/(y - z)$, independent of t.*

Proof: Four points z_1, z_2, z_3 and z_4 in the complex plane lie on a circle if and only if the cross-ratio

$$cr(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

is real. Since we compute

$$cr(x, x + ty, x + (s + t)z, x + syz/(y - z)) = \frac{s + t}{s}$$

the given points are on a circle. ■

Lay down complex coordinates with $S = 0$ and E and F on the positive real axis. Then there are real r_1, r_2 and R with $B = r_1 A$, $F = r_2 E$ and $D = E + R(A - E)$ and hence $AE/ED = BF/FC$ gives

$$C = F + R(B - F) = r_2(1 - R)E + r_1 RA.$$

The line CD consists of all points of the form $sC + (1 - s)D$ for real s. Since T lies on this line and has zero imaginary part, we see from $\mathrm{Im}(sC + (1 - s)D) = (sr_1 R + (1 - s)R)\mathrm{Im}(A)$ that it corresponds to $s = -1/(r_1 - 1)$. Thus

$$T = \frac{r_1 D - C}{r_1 - 1} = \frac{(r_2 - r_1)(R - 1)E}{r_1 - 1}.$$

Apply the lemma with $x = E$, $y = A - E$, $z = (r_2 - r_1)E/(r_1 - 1)$, and $s = (r_2 - 1)(r_1 - r_2)$. Setting $t = 1$ gives

$$(x, x + y, x + (s + 1)z) = (E, A, S = 0)$$

and setting $t = R$ gives

$$(x, x + Ry, x + (s + R)z) = (E, D, T).$$

Therefore the circumcircles to SAE and TDE meet at

$$x + \frac{syz}{y - z} = \frac{AE(r_1 - r_2)}{(1 - r_1)E - (1 - r_2)A} = \frac{AF - BE}{A + F - B - E}.$$

This last expression is invariant under simultaneously interchanging A and B and interchanging E and F. Therefore it is also the intersection of the circumcircles of SBF and TCF.

(This problem was proposed by Zuming Feng and Zhonghao Ye.)

2 Team Selection Test 2006

1. A communications network consisting of some terminals is called a *3-connector* if among any three terminals, some two of them can directly communicate with each other. A communications network contains a *windmill* with n blades if there exist n pairs of terminals $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$ such that each x_i can directly communicate with the corresponding y_i and there is a *hub* terminal that can directly communicate with each of the $2n$ terminals $x_1, y_1, \ldots, x_n, y_n$. Determine the minimum value of $f(n)$, in terms of n, such that a 3-connector with $f(n)$ terminals always contains a windmill with n blades.

Solution: The answer is

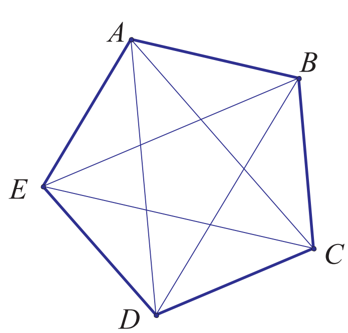
$$f(n) = \begin{cases} 6 & \text{if } n = 1; \\ 4n+1 & \text{if } n \geq 2. \end{cases}.$$

We will use *connected* as a synonym for directly communicating, call a set of k terminals for which each of the $\binom{k}{2}$ pairs of terminals is connected *complete* and call a set of $2k$ terminals forming k disjoint connected pairs a *k-matching*.

We first show that $f(n) = 4n+1$ for $n > 1$. The $4n$-terminal network consisting of two disconnected complete sets of $2n$ terminals clearly does not contain an n-bladed windmill (henceforth called an n-mill), since such a windmill requires a set of $2n+1$ connected terminals. So we need only demonstrate that $f(n) = 4n+1$ is sufficient.

Note that we can inductively create a k-matching in any subnetwork of $2k+1$ elements, as there is a connected pair in any set of three or more terminals. Also, the set of terminals that are not connected to a given terminal x must be complete, as otherwise there would be a set of three mutually disconnected terminals. We now proceed by contradiction and assume that there is a $(4n+1)$-terminal network without an n-mill. Any terminal x must then be connected to at least $2n$ terminals, for otherwise there would be a complete set of size at least $2n+1$, which includes an n-mill. In addition, x cannot be directly connected to more than $2n$ terminals, for otherwise we could construct an n-matching among these, and therefore an n-mill. Therefore every terminal is connected to precisely $2n$ others.

If we take two terminals u and v that are not connected we can then note that at least one must be connected to the $4n-1$ remaining terminals, and therefore there must be exactly one, w, to which both are connected. The rest of the network now consists of two complete sets of terminals A and B of size $2n-1$, where every terminal in A is connected to u and not connected to v, and every terminal in B is not connected to u and connected to v. If w were connected to any terminal in A or B, it would form a blade with this element and hub u or v respectively, and we could fill out the rest of an n-mill with terminals in A or B respectively. Hence w is only connected to two terminals, and therefore $n = 1$.



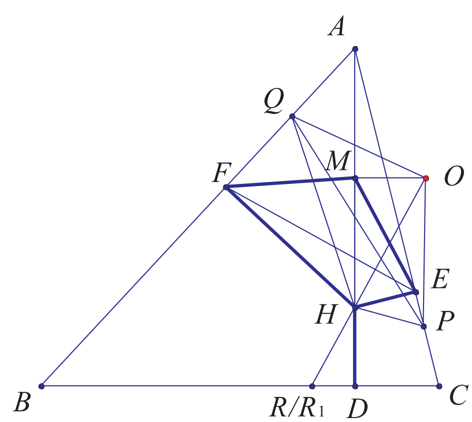
Examining the preceding proof, we can find the only 5-terminal network with no 1-mill: With terminals labeled A, B, C, D, and E, the connected pairs are (A, B), (B, C), (C, D), (D, E), and (E, A). (As indicated in the figure above, a pair of terminals are connected if and only if the edge connecting them are darkened.) To show that any 6-terminal network has a 1-mill, we note that any complete set of three terminals is a 1-mill. We again work by contradiction. Any terminal a would have to be connected to at least three others, b, c, and d, or the terminals not connected to a would form a 1-mill. But then one of the pairs (b, c), (c, d), and (b, d) must be connected, and this creates a 1-mill with that pair and a.

(This problem was proposed by Cecil C Rousseau.)

2. In acute triangle ABC, segments AD, BE, and CF are its altitudes, and H is its orthocenter. Circle ω, centered at O, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively. The circumcircle of triangle OPQ is tangent to segment BC at R. Prove that $CR/BR = ED/FD$.

Note: We present two solutions. We set $\angle CAB = x$, $\angle ABC = y$, and $\angle BCA = z$. Without loss of generality, we assume that Q is in between A and F. It is not difficult to show that P is in between C and E. (This is because $\angle FQH = \angle APH$.)

First Solution: (Based on work by Ryan Ko) Let M be the midpoint of segment AH. Since $\angle AEH = \angle AFH = 90°$, quadrilateral $AEHF$ is cyclic with M as its circumcenter. Hence triangle EFM is isosceles with vertex angle $\angle EMF = 2\angle CAB = 2x$. Likewise, triangle PQO is also an isosceles angle with vertex angle $\angle POQ = 2x$. Therefore, triangles EFM and PQO are similar.



Since $AEHF$ and $APHQ$ are cyclic, we have $\angle EFH = \angle EAH = \angle PQH$ and $\angle FEH = \angle FAH = \angle QPH$. Consequently, triangles HEF and HPQ are similar. It is not difficult to see that quadrilaterals $EHFM$ and $PHQO$ are similar. More precisely, if $\angle QHF = \theta$, there is a spiral similarity $\mathbf{S}$, centered at H with clockwise rotation angle θ and ratio QH/FH, that sends $FMEH$ to $QOPH$. Let R_1 be the point in between B and D such that $\angle R_1HD = \theta$. Then triangles QHF and R_1HD are similar. Hence $\mathbf{S}(D) = R_1$. It follows that

$$\mathbf{S}(DFME) = R_1QOP.$$

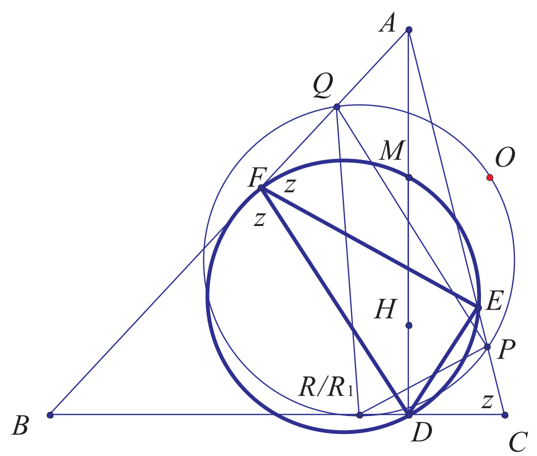
It is well known that points D, E, F, and M lie on a circle (the **nine-point circle** of triangle ABC). (This fact can be established easily by noting that $ABDE$ and $ACDF$ are cyclic, implying that

$\angle FDB = \angle CAF = x$, $\angle EDC = \angle BAE = x$, and $\angle EDF = 180^\circ - 2x = 180^\circ - \angle EMF$.) Since $DFME$ is cyclic, R_1QOP must also be cyclic. By the given conditions of the problem, we conclude that $R_1 = R$, implying that

$$\mathbf{S}(DEF) = RPQ,$$

or triangles DEF and RPQ are similar. It follows that

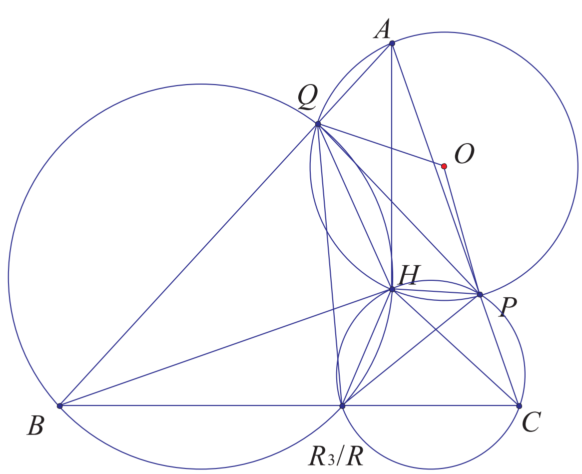
$$\frac{ED}{FD} = \frac{PR}{QR}.$$



Now we are ready to finish our proof. Since $ACDF$ and $ABDE$ are cyclic, $\angle BFD = \angle AFE = \angle ACB = z$. Thus $\angle DFE = 180^\circ - 2z$. Since triangles DEF and RPQ are similar, $\angle RQP = 180^\circ - 2z$. Because CR is tangent to the circumcircle of triangle PQR, $\angle CRP = \angle RQP = 180^\circ - 2z$. Thus, in triangle CPR, $\angle CPR = z$, and so it is isosceles with $CR = PR$. Likewise, we have $BR = QR$. Therefore, we have

$$\frac{ED}{FD} = \frac{PR}{QR} = \frac{CR}{BR}.$$

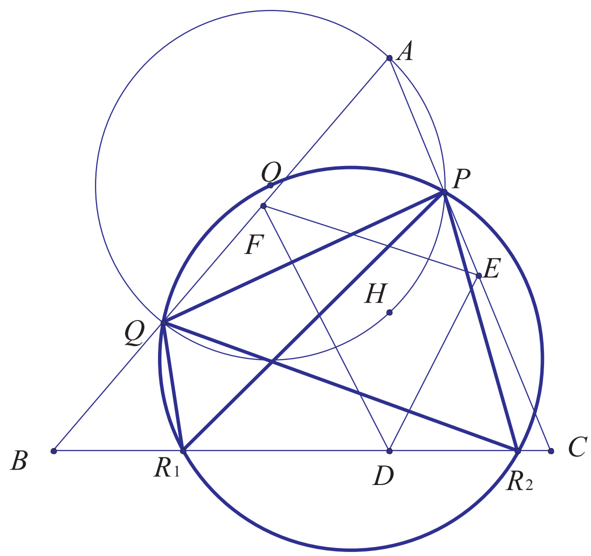
Second Solution: (Based on work by Zarathustra Brady) Let the circumcircle of triangle BQH meet line BC at R_3 (other than B).



Since $APHQ$ and $BQHR_3$ are cyclic, $\angle PHQ = 180^\circ - \angle PAQ$ and $\angle QHR_3 = 180^\circ - \angle QBR_3$, implying that $\angle PHR_3 = 360^\circ - \angle PHQ - \angle QHR_3 = 180^\circ - \angle ACB$. Hence $CPHR_3$ is also cyclic.

(We just established a special case of **Miquel's Theorem**.) Because $BQHR_3$ and CR_3HP are cyclic, we have $\angle QR_3H = \angle QBH = 90^\circ - \angle BAC$ and $\angle HR_3P = \angle HCP = 90^\circ - \angle BAC$. Hence $\angle QR_3P = 180^\circ - 2\angle BAC = 180^\circ - 2x$. Likewise, we have $\angle PQR = 180^\circ - 2z$ and $\angle R_3PQ = 180^\circ - 2y$. As we have shown in the first solution, triangle DEF have the same angles. Hence triangle R_3PQ is similar to triangle DEF. Also note that $\angle POQ + \angle PR_3Q = 2x + 180^\circ - 2x = 180^\circ$, implying that R_3 lies on the circumcircle of triangle OPQ. By the given condition, have $R_3 = R$. We can then finish our proof as we did in the first solution.

Note: As we have seen, the first solution is related to the 9-point circle of the triangle, and the second is related to the Miquel's theorem. Indeed, it is the special case (for $R_1 = R_2$) of the following interesting facts:



In acute triangle ABC, segments AD, BE, and CF are its altitudes, and H is its orthocenter. Circle ω, centered at O, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively.

(a) The perpendicular bisectors of segments BQ and CP meet at a point R_1 lying on line BC.

(b) There is a point R_2 on line BC such that triangle PQR_2 is similar to triangle EFD.

(c) Points O, P, Q, R_1, and R_2 are cyclic.

(This problem was proposed by Zuming Feng and Zhonghao Ye.)

3. Find the least real number k with the following property: if the real numbers x, y, and z are not all positive, then

$$k(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq (xyz)^2 - xyz + 1.$$

First Solution: The answer is $k = \frac{16}{9}$.

We start with a lemma.

Lemma 1. *If real numbers s and t are not all positive, then*

$$\frac{4}{3}(s^2 - s + 1)(t^2 - t + 1) \geq (st)^2 - st + 1. \qquad (*)$$

Proof: Without loss of generality, we assume that $s \geq t$.

We first assume that $s \geq 0 \geq t$. Setting $u = -t$, $(*)$ reads

$$\frac{4}{3}(s^2 - s + 1)(u^2 + u + 1) \geq (su)^2 + su + 1,$$

or

$$4(s^2 - s + 1)(u^2 + u + 1) \geq 3s^2u^2 + 3su + 3.$$

Expanding the left-hand side gives

$$4s^2u^2 + 4s^2u - 4su^2 - 4su + 4s^2 + 4u^2 - 4s + 4u + 4 \geq 3s^2u^2 + 3su + 3,$$

or

$$s^2u^2 + 4u^2 + 4s^2 + 1 + 4s^2u + 4u \geq 4su^2 + 4s + 7su$$

which is evident as $s^2u^2 + 4u^2 \geq 4su^2$, $4s^2 + 1 \geq 4s$, and $4s^2u + 4u \geq 8su \geq 7su$.

We second assume that $0 \geq s \geq t$. Let $v = -s$. By our previous argument, we have

$$\frac{4}{3}(v^2 - v + 1)(t^2 - t + 1) \geq (vt)^2 - vt + 1.$$

It is clear that $t^2 - t + 1 > 0$, $s^2 - s + 1 \geq v^2 - v + 1$, and $(vt)^2 - vt + 1 \geq (st)^2 - st + 1$. Combining the last four inequalities gives $(*)$, and this completes the proof of the lemma. ∎

Now we show that if x, y, z are not all positive real numbers, then

$$\frac{16}{9}(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq (xyz)^2 - xyz + 1. \qquad (**)$$

We consider three cases.

(a) We assume that $y \geq 0$. Setting $(s, t) = (y, z)$ and then $(s, t) = (x, yz)$ in the lemma gives the desired result.

(b) We assume that $0 \geq y$. Setting $(s, t) = (x, y)$ and then $(s, t) = (xy, z)$ in the lemma gives the desired result.

Finally, we confirm that the minimum value of k is $\frac{16}{9}$ by noting that the equality holds in $(**)$ when $(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

Second Solution: We establish $(**)$ by showing

$$g(z) = \frac{16}{9}(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) - (xyz)^2 + xyz - 1 \geq 0.$$

Note that $g(z)$ is a quadratic in z whose axis of symmetry (found by comparing the linear and quadratic terms) is at

$$\begin{aligned} z &= \frac{1}{2} - \frac{9}{32} \cdot \frac{xy}{(x^2 - x + 1)(y^2 - y + 1)} \\ &= \frac{1}{2} - \frac{9}{32} \cdot \frac{1}{\left(x + \frac{1}{x} - 1\right)\left(y + \frac{1}{y} - 1\right)}. \end{aligned}$$

For any t, we have $\left|x + \frac{1}{x} - 1\right| \geq 1$, so the absolute value of the second quantity on the right-hand side of the above equation is at most $\frac{9}{32}$, which is less than $\frac{1}{2}$. That is, the axis of symmetry occurs to the right side of the y-axis, so we only decrease the difference between the sides by replacing z by 0. But when $z = 0$, we only need to show

$$g(0) = \frac{16}{9}(x^2 - x + 1)(y^2 - y + 1) - 1 \geq 0,$$

which is evident as $t^2 - t + 1 = \left(t - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$.

Third Solution: This is the Calculus version of the second solution. We maintain the same notation as in the second solution. We have

$$\frac{dg}{dz} = \frac{16}{9}(2z - 1)(x^2 - x + 1)(y^2 - y + 1) - 2zx^2y^2 + xy$$

or

$$\frac{dg}{dz} = 2z\left[\frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1) - x^2y^2\right] + \left[xy - \frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1)\right]. \qquad (\dagger)$$

It is evident that

$$\frac{4}{3}(t^2 - t + 1) \geq t^2 \geq 0$$

as it is equivalent to $t^2 - 4t + 4 = (t - 2)^2 \geq 0$. It follows that

$$2z\left[\frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1) - x^2y^2\right] \leq 0;$$

that is, the first summand on the right-hand side of ($\dagger$) is not positive. It is also evident that

$$\frac{4}{3}(t^2 - t + 1) \geq t$$

as it is equivalent to $4t^2 - 7t + 4 = 4\left(t - \frac{7}{8}\right)^2 + \frac{15}{16} > 0$. If $y \geq 0$, then multiplying the inequalities

$$\frac{4}{3}(x^2 - x + 1) \geq x \geq 0 \quad \text{and} \quad \frac{4}{3}(y^2 - y + 1) \geq y \geq 0$$

gives

$$\frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1) - xy \geq 0.$$

If $y < 0$, then $xy < 0$, and so

$$\frac{4}{3}(x^2 - x + 1)\frac{4}{3}(y^2 - y + 1) \geq 0 \geq xy.$$

In either case, we have shown that the second summand in ($\dagger$) is also negative. We conclude that $\frac{dg}{dz} \leq 0$ for $z \leq 0$. Hence $g(z)$ reaches minimum when $z = 0$, and we can finish as we did in the second solution.

(This problem was proposed by Titu Andreescu and Gabriel Dospinescu.)

4. Let n be a positive integer. Find, with proof, the least positive integer d_n which cannot be expressed in the form

$$\sum_{i=1}^{n}(-1)^{a_i}2^{b_i},$$

where a_i and b_i are nonnegative integers for each i.

Solution: The answer is $d_n = (2^{2n+1}+1)/3$. We first show that d_n cannot be obtained. For any p let $t(p)$ be the minimum n required to express p in the desired form and call any realization of this minimum a *minimal representation*. If p is even, any sequence of b_i that can produce p must contain an even number of zeros. If this number is nonzero, then canceling one against another or replacing two with a $b_i = 1$ term would reduce the number of terms in the sum. Thus a minimal representation cannot contain a $b_i = 0$ term, and by dividing each term by two we see that $t(2m) = t(m)$. If p is odd, there must be at least one $b_i = 0$ and removing it gives a sequence that produces either $p-1$ or $p+1$. Hence

$$t(2m-1) = 1 + \min(t(2m-2), t(2m)) = 1 + \min(t(m-1), t(m)).$$

With d_n as defined above and $c_n = (2^{2n}-1)/3$, we have $d_0 = c_1 = 1$, so $t(d_0) = t(c_1) = 1$ and

$$t(d_n) = 1 + \min(t(d_{n-1}), t(c_n)) \quad \text{and} \quad t(c_n) = 1 + \min(t(d_{n-1}), t(c_{n-1})).$$

Hence, by induction, $t(c_n) = n$ and $t(d_n) = n+1$ and d_n cannot be obtained by a sum with n terms.

Next we show by induction on n that any positive integer less than d_n can be obtained with n terms. By the inductive hypothesis and symmetry about zero, it suffices to show that by adding one summand we can reach every p in the range $d_{n-1} \le p < d_n$ from an integer q in the range $-d_{n-1} < q < d_{n-1}$. Suppose that $c_n + 1 \le p \le d_n - 1$. By using a term 2^{2n-1}, we see that $t(p) \le 1 + t(|p - 2^{2n-1}|)$. Since $d_n - 1 - 2^{2n-1} = 2^{2n-1} - (c_n + 1) = d_{n-1} - 1$, it follows from the inductive hypothesis that $t(p) \le n$. Now suppose that $d_{n-1} \le p \le c_n$. By using a term 2^{2n-2}, we see that $t(p) \le 1 + t(|p - 2^{2n-2}|)$. Since $c_n - 2^{2n-2} = 2^{2n-2} - d_{n-1} = c_{n-1} < d_{n-1}$, it again follows that $t(p) \le n$.

(This problem was proposed by Richard Stong.)

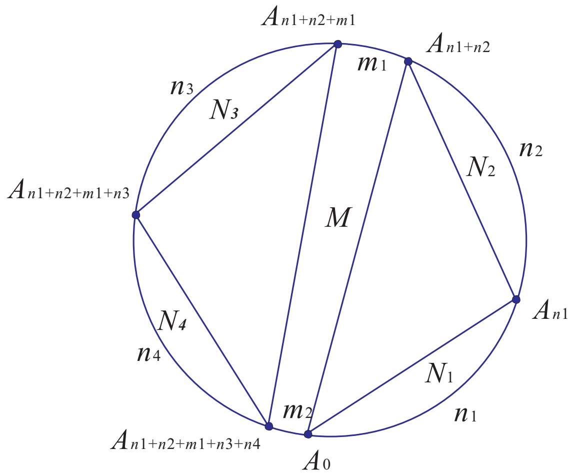
5. Let n be a given integer with n greater than 7, and let $\mathcal{P}$ be a convex polygon with n sides. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. A triangle in the triangulation of $\mathcal{P}$ is an interior triangle if all of its sides are diagonals of $\mathcal{P}$.

Express, in terms of n, the number of triangulations of $\mathcal{P}$ with exactly two interior triangles, in closed form.

Solution: The answer is

$$n2^{n-9}\binom{n-4}{4}.$$

Denote the vertices of P counter-clockwise by $A_0, A_1 \ldots, A_{n-1}$. We will count first the number of triangulations of P with two interior triangles positioned as in the following figure. We say that such a triangulation starts at A_0.



The numbers m_1, m_2, n_1, n_2, n_3, n_4 in the figure denote the number of sides of P determining the regions N_1, N_2, N_3, N_4 and M that consist of exterior triangles (triangles that are not interior). The two interior triangles are

$$A_0 A_{n_1} A_{n_1+n_2} \quad \text{and} \quad A_{n_1+n_2+m_1} A_{n_1+n_2+m_1+n_3} A_{n_1+n_2+m_1+n_3+n_4},$$

respectively.

We will show that triangulations starting at A_0 are in bijective correspondence to 7-tuples

$$(m, n_1, n_2, n_3, n_4, w_M, w_N),$$

where $m \geq 0$, $n_1, n_2, n_3, n_4 \geq 2$ are integers,

$$m + n_1 + n_2 + n_3 + n_4 = n, \qquad (\dagger)$$

w_M is a binary sequence (sequence of 0's and 1's) of length m and w_N is a binary sequence of length $n - m - 8$.

Indeed, given a triangulation as in the figure, the numbers $m = m_1 + m_2$ and n_1, n_2, n_3, n_4 satisfy (†) and the associated constraints.

Further, the triangulation of the outside region N_1 determines a binary sequence of length $n_1 - 2$ as follows. Denote the exterior triangle in N_1 using the diagonal $A_0 A_{n_1}$ by T_1. If $n_1 \geq 3$, T_1 has a unique neighboring exterior triangle in N_1, denoted T_2. If $n_1 \geq 4$, the triangle T_2 has another neighbor in N_1 denoted T_3, etc. Thus we have a sequence of $n_1 - 1$ exterior triangles in N_1. We encode this sequence as follows. If T_1 uses the vertex A_1 as its third vertex we encode this by 0 and if it uses A_{n_1-1} we encode this by 1. In each case there are two possible choices for the third vertex in T_2. If the one with smaller index is used we encode this by 0 and if the one with larger index is used we encode this by 1. Eventually, a sequence of $n_1 - 2$ 0's and 1's is constructed describing the choice of the third vertex in the triangles $T_1, \ldots, T_{n_1-2}$. Finally, there is only one choice for the third vertex in the triangle T_{n_1-1} (this triangle is uniquely determined by the previous one), so we get 2^{n_1-2} possible triangulations of N_1 encoded in a binary sequence of length $n_1 - 2$. Similarly, there are 2^{n_i-2} triangulations of the region N_i, $i = 1, 2, 3, 4$, encoded by binary sequences of length $n_i - 2$. Thus a binary sequence w_N of length $n_1 - 2 + n_2 - 2 + n_3 - 2 + n_4 - 2 = n - m - 8$, uniquely determines the triangulations of the regions N_1, N_2, N_3, N_4 (once the regions are precisely determined within P, which is done once m_1,m_2,n_1,n_2,n_3 and n_4 are known).

It remains to uniquely encode the triangulation of the middle region M. Denote by M_1 the unique exterior triangle in M using the diagonal $A_0A_{n_1+n_2}$. If $m \geq 2$. M_1 has a unique neighboring exterior triangle M_2 in M. If $m \geq 3$, the triangle M_2 has another neighbor in M denoted M_3, etc. Thus we have a sequence of m exterior triangles in M. We encode this sequence as follows. If M_1 uses the vertex $A_{n_1+n_2+1}$ as its third vertex we encode this by 0 and if it uses A_{n-1} we encode this by 1. In each case there are two possible choices for the third vertex in M_2. If the one with smaller index is used we encode this by 0 and if the one with larger index is used we encode this by 1. Eventually, a sequence of m 0's and 1's is constructed describing the choice of the third vertex in the triangles $M_1, \ldots, M_m$. Thus a binary sequence w_M of length m uniquely determines the triangulation of the region M. In addition such a sequence w_M uniquely determines m_1 and m_2 as the number of 0's and 1's respectively in w_M and therefore also the exact position of the middle region M within P (once n_1 and n_2 are known), which in turn then exactly determines the position of all the regions considered in the figure.

The number of solutions of the equation (†) subject to the given constraints is equal to the number of positive integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = n - 3,$$

which is $\binom{n-4}{4}$ (a sequence of $n-3$ objects is split into 5 nonempty groups by placing 4 separators in the $n-4$ available positions between the objects). Thus the number of 7-tuples $(m, n_1, n_2, n_3, n_4, w_M, w_N)$ describing triangulations as in the figure is

$$2^m \cdot 2^{n-m-8}\binom{n-4}{4} = 2^{n-8}\binom{n-4}{4}.$$

Finally, in order to get the total number of triangulations we multiply the above number by n (since we could start building the triangulation at any vertex rather than at A_0) and divide by 2 (since every triangulation is now counted twice, once as starting at one of the interior triangles and once as starting at the other).

Note: The problem is more trickier than it might seem. In particular, the idea of choosing m first and then letting the bits in w_M split it into m_1 and m_2 while, in the same time, determining the triangulation of M is not that obvious. If one does the "more natural thing" and chooses all the the numbers m_1, m_2, n_1, n_2, n_3, n_4 first and then tries to encode the triangulations of the obtained regions one gets into more complicated considerations involving the middle region M (and most likely has to resort to messy summations over different pairs m_1, m_2).

As an quick exercise, one can compute number of triangulations of P ($n \geq 6$) with exactly one interior region. This is much easier since there is no middle region M to worry about and the number of triangulations is

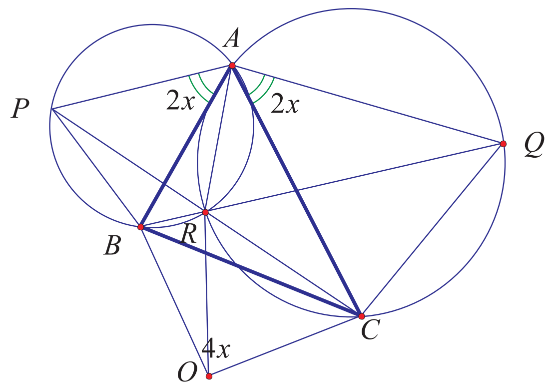
$$\frac{n}{3}2^{n-6}\binom{n-4}{2}.$$

(This problem was proposed by Zoran Sunik.)

6. Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.

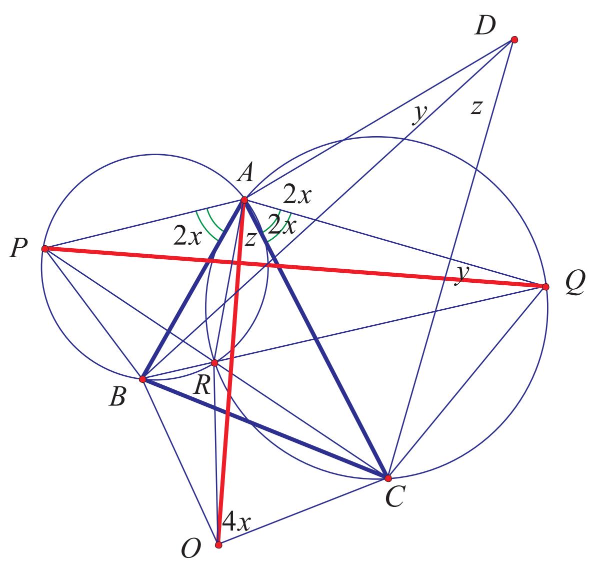
Note: We present five differen approaches. The first three synthetic solutions are all based on the following simple observation.

We first note that $APBR$ and $AQCR$ are cyclic quadrilaterals. It is easy to see that triangles APC and ABQ are congruent to each other, implying that $\angle APR = \angle APC = \angle ABQ = \angle ABR$. Thus, $APBR$ is a cyclic quadrilateral. Likewise, we can show that $AQCR$ is also cyclic.



Let $\angle PAB = 2x$. Then in isosceles triangle APB, $\angle APB = 90^\circ - x$. In cyclic quadrilateral $APBR$, $\angle ARB = 180^\circ - \angle APB = 90^\circ + x$. Likewise, $\angle ARC = 90^\circ + x$. Hence $\angle BRC = 360^\circ - \angle ARB - \angle ARC = 180^\circ - 2x$. It follows that $\angle BOC = 4x$.

First Solution: Reflect C across line AQ to D. Then $\angle BAD = 4x + \angle BAC = \angle BAQ$. It is easy to see that triangles BAD and PAQ are congruent, implying that $\angle ADB = \angle AQP = y$.



Note also that CAD and COB are two isosceles triangles with the same vertex angle, and so they are similar to each other. It follows that triangle CAO and CBD are similar by SAS (side-angle-side), implying that $\angle CAO = \angle CDB = z$.

The angle formed by lines AO and PQ is equal to

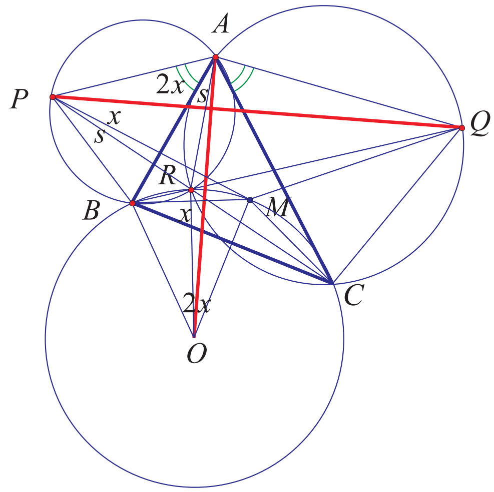
$$180^\circ - \angle OAQ - \angle AQP = 180^\circ - \angle OAC - \angle CAQ - \angle AQP = 180^\circ - z - 2x - y.$$

Since AQ is perpendicular to the base CD in isosceles triangle ACD, we have

$$90^\circ = \angle QAD + \angle CDA = \angle QAD + \angle ADB + \angle BDC = 2x + y + z.$$

Combining the last two equations yields that fact the angle formed by lines AO and PQ is equal to 90°; that is, $AO \perp PQ$.

Second Solution: We maintain the same notations as in the first solution. Let M be the midpoint of arc $\widehat{BC}$ on the circumcircle of triangle BOC. Then $BM = CM$. Since triangles APC and ABQ are congruent, $PC = BQ$. Since $BRMC$ is cyclic, $\angle PCM = \angle RCM = \angle RBM = \angle QBM$. Hence triangles BMQ and CMP are congruent by SAS. It follows triangles MPQ and MBC are similar. Since $\angle BOC = 4x$, $\angle MBC = \angle MCB = x$, and so $\angle MPQ = x$.



Note that both triangles PAB and MOB are isosceles triangles with vertex angle $2x$; that is, they are similar to each other. Hence triangles BMP and BOA are also similar by SAS, implying that $\angle OAB = MPB = s$. We also note that in isosceles triangle APB,

$$90^\circ = \angle APB + \angle PAB/2 = \angle APQ + \angle QPM + \angle MPB + \angle PAB/2 = \angle APQ + 2x + s.$$

Putting the above together, we conclude that

$$\angle PAO + \angle APQ = \angle PAB + \angle BAO + \angle APQ = 2x + s + \angle APQ = 90^\circ,$$

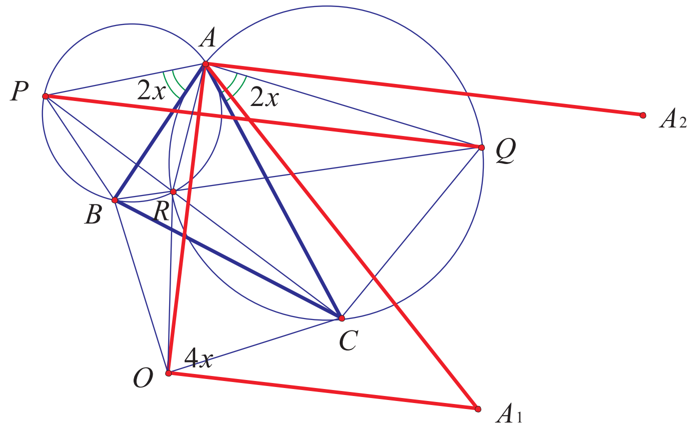
that is $AO \perp PQ$.

Third Solution: We consider two rotations:

$$\mathbf{R}_1: \quad \text{a counterclockwise } 2x \text{ (degree) rotation centered at } A,$$
$$\mathbf{R}_2: \quad \text{a clockwise } 4x \text{ (degree) rotation centered at } O.$$

Let $\mathbf{T}$ denote the composition $\mathbf{R}_1\mathbf{R}_2\mathbf{R}_1$. Then $\mathbf{T}$ is a counterclockwise $2x - 4x + 2x = 0^\circ$ rotation; that is, $\mathbf{T}$ is translation. Note that

$$\mathbf{T}(P) = \mathbf{R}_1(\mathbf{R}_2(\mathbf{R}_1(P))) = \mathbf{R}_1(\mathbf{R}_2(B)) = \mathbf{R}_1(C) = Q,$$



or, $\mathbf{T}$ is the vector translation $\overrightarrow{PQ}$.

Let $A_1 = \mathbf{R}_2(A)$ and $A_2 = \mathbf{R}_1(A_1)$. Then $\mathbf{T}(A) = A_2$; that is, $\overrightarrow{AA_2} = \overrightarrow{PQ}$, or $AA_2 \parallel PQ$.

By the definitions of $\mathbf{R}_2$ and $\mathbf{R}_1$, we know that triangles OAA_1 and A_1AA_2 are isosceles triangles with respect vertex angles $\angle AOA_1 = 4x$ and $\angle A_1AA_2 = 2x^\circ$. It is routine to compute that $\angle OAA_2 = 90^\circ$; that $AO \perp AA_2$, or $AO \perp PQ$.

Fourth Solution: (By Ian Le) In this solutions, let each lowercase letter denote the number assigned to the point labeled with the corresponding uppercase letter. We further assume that A is origin; that is, let $a = 0$. Let $\omega = e^{2xi}$ (or $\omega = \cos(2x) + i\sin(2x)$, and $\omega^{-1} = \cos(2x) - i\sin(2x)$). Then because O lies on the perpendicular bisector of BC and $\angle BOC = 4x$,

$$o = c + \frac{(b-c)i}{2\omega\sin(2x)} = c + \frac{bi}{2\omega\sin(2x)} - \frac{ci}{2\omega\sin(2x)}.$$

Note that

$$c - \frac{ci}{2\omega\sin(2x)} = c + \frac{c\omega^{-1}}{2i\sin(2x)} = \frac{c(\omega^{-1} + 2i\sin(2x))}{2i\sin(2x)} = \frac{c\omega}{2i\sin(2x)},$$

Combining the last two equations gives

$$o = \frac{bi}{2\omega\sin(2x)} + \frac{c\omega}{2i\sin(2x)} = -\frac{b}{2i\omega\sin(2x)} + \frac{c\omega}{2i\sin(2x)} = \frac{1}{2i\sin(2x)}\left(c\omega - \frac{b}{\omega}\right).$$

Now we note that $p = \frac{b}{\omega}$ and $q = c\omega$. Consequently, we obtain

$$\frac{q-p}{o-a} = 2i\sin(2x),$$

which is clearly a pure imaginary number; that is, $OA \perp PQ$.

Fifth Solution: (By Lan Le) In this solutions, we set $BC = a, AB = c$, $CA = b$, $A = \angle BAC$, $B = \angle ABC$, and $C = \angle BCA$. We use the fact that

$$OA \perp PQ \quad \text{if and only if} \quad AP^2 - AQ^2 = OP^2 - OQ^2.$$

Clearly $AP^2 - AQ^2 = c^2 - b^2$. It remains to show that

$$OP^2 - OQ^2 = c^2 - b^2. \tag{$*$}$$

In isosceles triangles APB and BOC, $BP = 2c\sin x$ and $BO = \frac{a}{2\sin(2x)}$. Note that $\angle PBA + \angle ABC + \angle CBO = 90° - x + B + 90° - 2x = 180° + B - 3x$. Applying the **law of cosines** to triangle PBO yields

$$OP^2 = 4c^2\sin^2 x + \frac{a^2}{4\sin^2(2x)} + \frac{ac\cos(B-3x)}{\cos x}.$$

In exactly the same way, we can show that

$$OQ^2 = 4b^2\sin^2 x + \frac{a^2}{4\sin^2(2x)} + \frac{ab\cos(C-3x)}{\cos x}.$$

Hence

$$OP^2 - OQ^2 = 4(c^2-b^2)\sin^2 x + \frac{a}{\cos x}(c\cos(B-3x) - b\cos(C-3x)). \qquad (\dagger)$$

Using **Addition and Substraction formulas** and the **law of sines** (more precisely, $c\sin B = b\sin C$), we have

$$\begin{aligned} & c\cos(B-3x) - b\cos(C-3x) \\ = \; & c\cos(3x)\cos B + c\sin(3x)\sin B - b\cos(3x)\cos C - b\sin(3x)\sin C \\ = \; & \cos(3x)(c\cos B - b\cos C). \end{aligned}$$

Substituting the last equation into ($\dagger$) gives

$$OP^2 - OQ^2 = 4(c^2-b^2)\sin^2 x + \frac{\cos 3x}{\cos x}(ac\cos B - ab\cos C).$$

Note that

$$ac\cos B - ab\cos C = c(a\cos B + b\cos A) - b(a\cos C + c\cos A) = c^2 - b^2.$$

Combining the last equations gives

$$OP^2 - OQ^2 = (c^2-b^2)\left(4\sin^2 x + \frac{\cos 3x}{\cos x}\right).$$

By the **Triple-angle formulas**, we have $\cos 3x = 4\cos^3 x - 3\cos x$, and so

$$OP^2 - OQ^2 = (c^2-b^2)(4\sin^2 x + 4\cos^2 x - 3) = c^2 - b^2,$$

which is ($*$).

(This problem was proposed by Zuming Feng and Zhonghao Ye.)

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1. Let n be a positive integer. Define a sequence by setting $a_1 = n$ and, for each $k > 1$, letting a_k be the unique integer in the range $0 \le a_k \le k-1$ for which $a_1 + a_2 + \cdots + a_k$ is divisible by k. For instance, when $n = 9$ the obtained sequence is $9, 1, 2, 0, 3, 3, 3, \ldots$. Prove that for any n the sequence $a_1, a_2, a_3, \ldots$ eventually becomes constant.

First Solution: For $k \ge 1$, let

$$s_k = a_1 + a_2 + \cdots + a_k.$$

We have

$$\frac{s_{k+1}}{k+1} < \frac{s_{k+1}}{k} = \frac{s_k + a_{k+1}}{k} \le \frac{s_k + k}{k} = \frac{s_k}{k} + 1.$$

On the other hand, for each k, s_k/k is a positive integer. Therefore

$$\frac{s_{k+1}}{k+1} \le \frac{s_k}{k},$$

and the sequence of quotients s_k/k is eventually constant. If $s_{k+1}/(k+1) = s_k/k$, then

$$a_{k+1} = s_{k+1} - s_k = \frac{(k+1)s_k}{k} - s_k = \frac{s_k}{k},$$

showing that the sequence a_k is eventually constant as well.

Second Solution: For $k \ge 1$, let

$$s_k = a_1 + a_2 + \cdots + a_k \qquad \text{and} \qquad \frac{s_k}{k} = q_k.$$

Since $a_k \le k - 1$, for $k \ge 2$, we have

$$s_k = a_1 + a_2 + a_3 + \cdots + a_k \le n + 1 + 2 + \cdots + (k-1) = n + \frac{k(k-1)}{2}.$$

Let m be a positive integer such that $n \le \frac{m(m+1)}{2}$ (such an integer clearly exists). Then

$$q_m = \frac{s_m}{m} \le \frac{n}{m} + \frac{m-1}{2} \le \frac{m+1}{2} + \frac{m-1}{2} = m.$$

We claim that

$$q_m = a_{m+1} = a_{m+2} = a_{m+3} = a_{m+4} = \ldots \, .$$

This follows from the fact that the sequence $a_1, a_2, a_3, \ldots$ is uniquely determined and choosing $a_{m+i} = q_m$, for $i \ge 1$, satisfies the range condition

$$0 \le a_{m+i} = q_m \le m \le m + i - 1,$$

and yields

$$s_{m+i} = s_m + iq_m = mq_m + iq_m = (m+i)q_m.$$

Third Solution: For $k \ge 1$, let

$$s_k = a_1 + a_2 + \cdots + a_k.$$

We claim that for some m we have $s_m = m(m-1)$. To this end, consider the sequence which computes the differences between s_k and $k(k-1)$, i.e., whose k-th term is $s_k - k(k-1)$. Note that the first term of this sequence is positive (it is equal to n) and that its terms are strictly decreasing since

$$(s_k - k(k-1)) - (s_{k+1} - (k+1)k) = 2k - a_{k+1} \geq 2k - k = k \geq 1.$$

Further, a negative term cannot immediately follow a positive term. Suppose otherwise, namely that $s_k > k(k-1)$ and $s_{k+1} < (k+1)k$. Since s_k and s_{k+1} are divisible by k and $k+1$, respectively, we can tighten the above inequalities to $s_k \geq k^2$ and $s_{k+1} \leq (k+1)(k-1) = k^2 - 1$. But this would imply that $s_k > s_{k+1}$, a contradiction. We conclude that the sequence of differences must eventually include a term equal to zero.

Let m be a positive integer such that $s_m = m(m-1)$. We claim that

$$m - 1 = a_{m+1} = a_{m+2} = a_{m+3} = a_{m+4} = \dots .$$

This follows from the fact that the sequence $a_1, a_2, a_3, \dots$ is uniquely determined and choosing $a_{m+i} = m - 1$, for $i \geq 1$, satisfies the range condition

$$0 \leq a_{m+i} = m - 1 \leq m + i - 1,$$

and yields

$$s_{m+i} = s_m + i(m-1) = m(m-1) + i(m-1) = (m+i)(m-1).$$

(This problem was suggested by Sam Vandervelde.)

2. A square grid on the Euclidean plane consists of all points (m, n), where m and n are integers. Is it possible to cover all grid points by an infinite family of discs with non-overlapping interiors if each disc in the family has radius at least 5?

Solution: It is not possible. The proof is by contradiction. Suppose that such a covering family $\mathcal{F}$ exists. Let $D(P, \rho)$ denote the disc with center P and radius ρ. Start with an arbitrary disc $D(O, r)$ that does not overlap any member of $\mathcal{F}$. Then $D(O, r)$ covers no grid point. Take the disc $D(O, r)$ to be maximal in the sense that any further enlargement would cause it to violate the non-overlap condition. Then $D(O, r)$ is tangent to at least three discs in $\mathcal{F}$. Observe that there must be two of the three tangent discs, say $D(A, a)$ and $D(B, b)$, such that $\angle AOB \leq 120°$. By the Law of Cosines applied to triangle ABO,

$$(a+b)^2 \leq (a+r)^2 + (b+r)^2 + (a+r)(b+r),$$

which yields

$$ab \leq 3(a+b)r + 3r^2, \qquad \text{and thus} \qquad 12r^2 \geq (a-3r)(b-3r).$$

Note that $r < 1/\sqrt{2}$ because $D(O, r)$ covers no grid point, and $(a-3r)(b-3r) \geq (5-3r)^2$ because each disc in $\mathcal{F}$ has radius at least 5. Hence $2\sqrt{3}r \geq (5-3r)$, which gives $5 \leq (3+2\sqrt{3})r < (3+2\sqrt{3})/\sqrt{2}$ and thus $5\sqrt{2} < 3 + 2\sqrt{3}$. Squaring both sides of this inequality yields $50 < 21 + 12\sqrt{3} < 21 + 12 \cdot 2 = 45$. This contradiction completes the proof.

Remark: The above argument shows that no covering family exists where each disc has radius greater than $(3 + 2\sqrt{3})/\sqrt{2} \approx 4.571$. In the other direction, there exists a covering family in which each disc has radius $\sqrt{13}/2 \approx 1.802$. Take discs with this radius centered at points of the form $(2m + 4n + \frac{1}{2}, 3m + \frac{1}{2})$, where m and n are integers. Then any grid point is within $\sqrt{13}/2$ of one of the centers and the distance between any two centers is at least $\sqrt{13}$. The extremal radius of a covering family is unknown.

(This problem was suggested by Gregory Galperin.)

3. Let S be a set containing n^2+n-1 elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

Solution: In order to apply induction, we generalize the result to be proved so that it reads as follows:

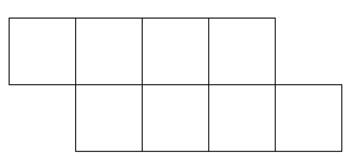
Proposition. If the n-element subsets of a set S with $(n+1)m-1$ elements are partitioned into two classes, then there are at least m pairwise disjoint sets in the same class.

Proof: Fix n and proceed by induction on m. The case of $m = 1$ is trivial. Assume $m > 1$ and that the proposition is true for $m-1$. Let $\mathcal{P}$ be the partition of the n-element subsets into two classes. If all the n-element subsets belong to the same class, the result is obvious. Otherwise select two n-element subsets A and B from different classes so that their intersection has maximal size. It is easy to see that $|A \cap B| = n-1$. (If $|A \cap B| = k < n-1$, then build C from B by replacing some element not in $A \cap B$ with an element of A not already in B. Then $|A \cap C| = k+1$ and $|B \cap C| = n-1$ and either A and C or B and C are in different classes.) Removing $A \cup B$ from S, there are $(n+1)(m-1)-1$ elements left. On this set the partition induced by $\mathcal{P}$ has, by the inductive hypothesis, $m-1$ pairwise disjoint sets in the same class. Adding either A or B as appropriate gives m pairwise disjoint sets in the same class. ■

Remark: The value n^2+n-1 is sharp. A set S with n^2+n-2 elements can be split into a set A with n^2-1 elements and a set B of $n-1$ elements. Let one class consist of all n-element subsets of A and the other consist of all n-element subsets that intersect B. Then neither class contains n pairwise disjoint sets.

(This problem was suggested by András Gyárfás.)

4. An *animal* with n *cells* is a connected figure consisting of n equal-sized square cells.[1] The figure below shows an 8-cell animal.



[1] Animals are also called *polyominoes*. They can be defined inductively. Two cells are *adjacent* if they share a complete edge. A single cell is an animal, and given an animal with n-cells, one with $n+1$ cells is obtained by adjoining a new cell by making it adjacent to one or more existing cells.

A *dinosaur* is an animal with at least 2007 cells. It is said to be *primitive* if its cells cannot be partitioned into two or more dinosaurs. Find with proof the maximum number of cells in a primitive dinosaur.

First Solution: Let s denote the minimum number of cells in a dinosaur; the number this year is $s = 2007$.

Claim: The maximum number of cells in a primitive dinosaur is $4(s-1)+1 = 8025$.

First, a primitive dinosaur can contain up to $4(s-1)+1$ cells. To see this, consider a dinosaur in the form of a cross consisting of a central cell and four arms with $s-1$ cells apiece. No connected figure with at least s cells can be removed without disconnecting the dinosaur.

The proof that no dinosaur with at least $4(s-1)+2$ cells is primitive relies on the following result.

Lemma Let D be a dinosaur having at least $4(s-1)+2$ cells, and let R (red) and B (black) be two complementary animals in D, i.e., $R \cap B = \varnothing$ and $R \cup B = D$. Suppose $|R| \le s-1$. Then R can be augmented to produce animals $\tilde{R} \supset R$ and $\tilde{B} = D \setminus \tilde{R}$ such that at least one of the following holds:

(i) $|\tilde{R}| \ge s$ and $|\tilde{B}| \ge s$,

(ii) $|\tilde{R}| = |R| + 1$,

(iii) $|R| < |\tilde{R}| \le s-1$.

Proof: If there is a black cell adjacent to R that can be made red without disconnecting B, then (ii) holds. Otherwise, there is a black cell c adjacent to R whose removal disconnects B. Of the squares adjacent to c, at least one is red, and at least one is black, otherwise B would be disconnected. Then there are at most three resulting components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of B after the removal of c. Without loss of generality, $\mathcal{C}_3$ is the largest of the remaining components. (Note that $\mathcal{C}_1$ or $\mathcal{C}_2$ may be empty.) Now $\mathcal{C}_3$ has at least $\lceil (3s-2)/3 \rceil = s$ cells. Let $\tilde{B} = \mathcal{C}_3$. Then $|\tilde{R}| = |R| + |\mathcal{C}_1| + |\mathcal{C}_2| + 1$. If $|\tilde{B}| \le 3s-2$, then $|\tilde{R}| \ge s$ and (i) holds. If $|\tilde{B}| \ge 3s-1$ then either (ii) or (iii) holds, depending on whether $|\tilde{R}| \ge s$ or not. ■

Starting with $|R| = 1$, repeatedly apply the Lemma. Because in alternatives (ii) and (iii) $|R|$ increases but remains less than s, alternative (i) eventually must occur. This shows that no dinosaur with at least $4(s-1)+2$ cells is primitive.

Second Solution: (Based on Andrew Geng's solution) Let $s = 2007$. We claim that the answer is $4s - 3 = 8025$.

Consider a graph with the cells as the vertices and whose edges connect adjacent cells. Let T be a spanning tree in this graph. By removing any vertex of T, we obtain at most four connected components, which we call the *limbs* of the vertex. Limbs with at least s vertices are called *big*. Suppose that every vertex of T contains a big limb, then consider a walk on T starting from an arbitrary vertex and always moving along the edge towards a big limb. Since T is a finite tree, this walk must traverse back on some edge at some point. Then the two connected components of T made by deleting this edge are both big, so they both contain at least s vertices, which means that the dinosaur is not primitive. It follows that a primitive dinosaur contains some vertex with no big limbs. By removing this vertex, we get at most four connected components with at most $s-1$ vertices each. This not only shows that a primitive dinosaur has at most $4s-3$ cells, but also shows that any such dinosaur consists of four limbs of $s-1$ cells each connected to a central cell. It is easy to see that such a dinosaur indeed exists.

(This problem was suggested by Reid Barton.)

5. Prove that for every nonnegative integer n, the number $7^{7^n}+1$ is the product of at least $2n+3$ (not necessarily distinct) primes.

Solution: The proof is by induction. The base is provided by the $n=0$ case, where $7^{7^0}+1 = 7^1+1 = 2^3$. To prove the inductive step, it suffices to show that if $x = 7^{2m-1}$ for some positive integer m then $(x^7+1)/(x+1)$ is composite. As a consequence, x^7+1 has at least two more prime factors than does $x+1$. To confirm that $(x^7+1)/(x+1)$ is composite, observe that

$$\begin{aligned}\frac{x^7+1}{x+1} &= \frac{(x+1)^7-((x+1)^7-(x^7+1))}{x+1}\\ &= (x+1)^6 - \frac{7x(x^5+3x^4+5x^3+5x^2+3x+1)}{x+1}\\ &= (x+1)^6 - 7x(x^4+2x^3+3x^2+2x+1)\\ &= (x+1)^6 - 7^{2m}(x^2+x+1)^2\\ &= \{(x+1)^3-7^m(x^2+x+1)\}\{(x+1)^3+7^m(x^2+x+1)\}\end{aligned}$$

Also each factor exceeds 1. It suffices to check the smaller one; $\sqrt{7x}\le x$ gives

$$\begin{aligned}(x+1)^3-7^m(x^2+x+1) &= (x+1)^3-\sqrt{7x}(x^2+x+1)\\ &\ge x^3+3x^2+3x+1-x(x^2+x+1)\\ &= 2x^2+2x+1\ge 113>1.\end{aligned}$$

Hence $(x^7+1)/(x+1)$ is composite and the proof is complete.

(This problem was suggested by Titu Andreescu.)

6. Let ABC be an acute triangle with ω, Ω, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to Ω at A and tangent externally to ω. Circle Ω_A is tangent internally to Ω at A and tangent internally to ω. Let P_A and Q_A denote the centers of ω_A and Ω_A, respectively. Define points P_B, Q_B, P_C, Q_C analogously. Prove that

$$8P_AQ_A\cdot P_BQ_B\cdot P_CQ_C\le R^3,$$

with equality if and only if triangle ABC is equilateral.

Solution: Let the incircle touch the sides AB, BC, and CA at C_1, A_1, and B_1, respectively. Set $AB=c$, $BC=a$, $CA=b$. By equal tangents, we may assume that $AB_1 = AC_1 = x$, $BC_1 = BA_1 = y$, and $CA_1 = CB_1 = z$. Then $a=y+z$, $b=z+x$, $c=x+y$. By the AM-GM inequality, we have $a \geq 2\sqrt{yz}$, $b \geq 2\sqrt{zx}$, and $c \geq 2\sqrt{xy}$. Multiplying the last three inequalities yields

$$abc \geq 8xyz, \qquad (\dagger),$$

with equality if and only if $x = y = z$; that is, triangle ABC is equilateral.

Let k denote the area of triangle ABC. By the Extended Law of Sines, $c = 2R \sin \angle C$. Hence

$$k = \frac{ab \sin \angle C}{2} = \frac{abc}{4R} \quad \text{or} \quad R = \frac{abc}{4k}. \qquad (\ddagger)$$

We are going to show that

$$P_A Q_A = \frac{xa^2}{4k}. \qquad (*)$$

In exactly the same way, we can also establish its cyclic analogous forms

$$P_B Q_B = \frac{yb^2}{4k} \quad \text{and} \quad P_C Q_C = \frac{zc^2}{4k}.$$

Multiplying the last three equations together gives

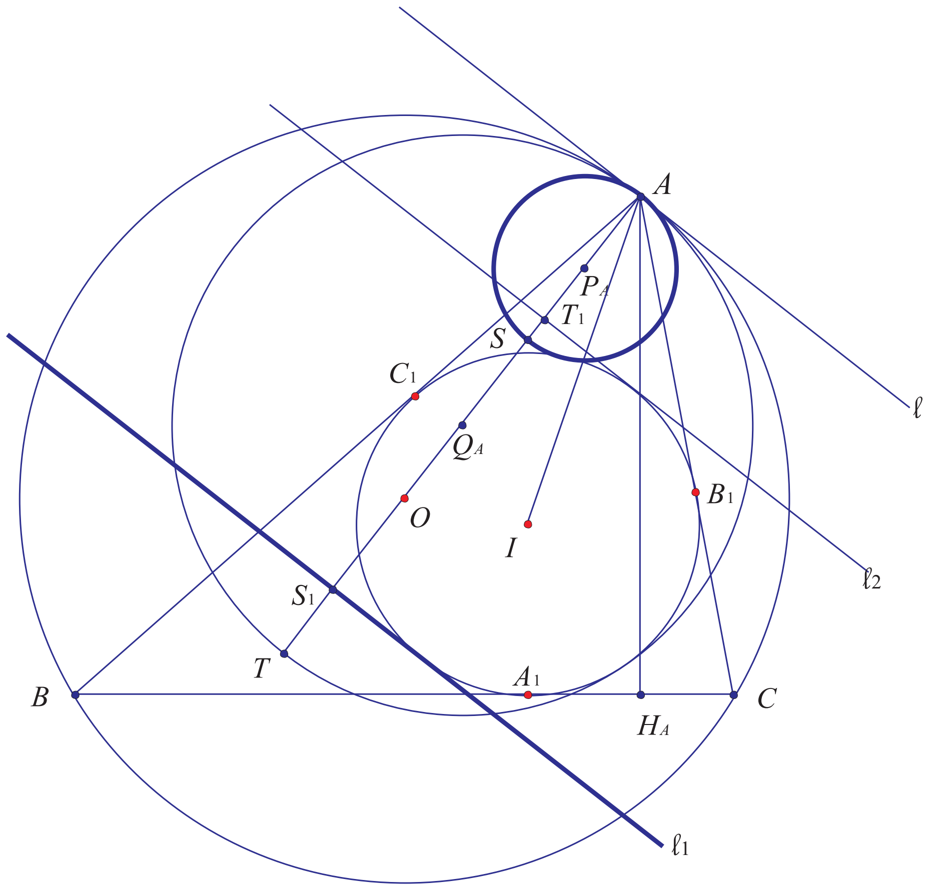
$$P_A Q_A \cdot P_B Q_B \cdot P_C Q_C = \frac{xyza^2b^2c^2}{64k^3}.$$

Further considering ($\dagger$) and ($\ddagger$), we have

$$8P_A Q_A \cdot P_B Q_B \cdot P_C QC = \frac{8xyza^2b^2c^2}{64k^3} \leq \frac{a^3b^3c^3}{64k^3} = R^3,$$

with equality if and only if triangle ABC is equilateral.

Hence it suffices to show ($*$). Let r, r_A, r'_A denote the radii of $\omega, \omega_A, \Omega_A$, respectively. We consider the inversion $\mathbf{I}$ with center A and radius x. Clearly, $\mathbf{I}(B_1) = B_1$, $\mathbf{I}(C_1) = C_1$, and $\mathbf{I}(\omega) = \omega$. Let ray AO intersect ω_A and Ω_A at S and T, respectively. It is not difficult to see that $AT > AS$, because ω is tangent to ω_A and Ω_A externally and internally, respectively. Set $S_1 = \mathbf{I}(S)$ and $T_1 = \mathbf{I}(T)$. Let ℓ denote the line tangent to Ω at A. Then the image of ω_A (under the inversion) is the line (denoted by ℓ_1) passing through S_1 and parallel to ℓ, and the image of Ω_A is the line (denoted by ℓ_2) passing through T_1 and parallel to ℓ. Furthermore, since ω is tangent to both ω_A and Ω_A, ℓ_1 and ℓ_2 are also tangent to the image of ω, which is ω itself. Thus the distance between these two lines is $2r$; that is, $S_1T_1 = 2r$. Hence we can consider the following configuration. (The darkened circle is ω_A, and its image is the darkened line ℓ_1.)



By the definition of inversion, we have $AS_1 \cdot AS = AT_1 \cdot AT = x^2$. Note that $AS = 2r_A$, $AT = 2r'_A$, and $S_1T_1 = 2r$. We have

$$r_A = \frac{x^2}{2AS_1}. \quad \text{and} \quad r'_A = \frac{x^2}{2AT_1} = \frac{x^2}{2(AS_1 - 2r)}.$$

Hence

$$P_AQ_A = AQ_A - AP_A = r'_A - r_A = \frac{x^2}{2}\left(\frac{1}{AS_1 - 2r} + \frac{1}{AS_1}\right).$$

Let H_A be the foot of the perpendicular from A to side BC. It is well known that $\angle BAS_1 = \angle BAO = 90^\circ - \angle C = \angle CAH_A$. Since ray AI bisects $\angle BAC$, it follows that rays AS_1 and AH_A are symmetric with respect to ray AI. Further note that both line ℓ_1 (passing through S_1) and line BC (passing through H_A) are tangent to ω. We conclude that $AS_1 = AH_A$. In light of this observation and using the fact $2k = AH_A \cdot BC = (AB + BC + CA)r$, we can compute P_AQ_A as follows:

$$\begin{aligned} P_AQ_A &= \frac{x^2}{2}\left(\frac{1}{AH_A - 2r} - \frac{1}{AH_A}\right) = \frac{x^2}{4k}\left(\frac{2k}{AH_A - 2r} - \frac{2k}{AH_A}\right) \\ &= \frac{x^2}{4k}\left(\frac{1}{\frac{1}{BC} - \frac{2}{AB+BC+CA}} - BC\right) = \frac{x^2}{4k}\left(\frac{1}{\frac{1}{y+z} - \frac{1}{x+y+z}} - (y+z)\right) \\ &= \frac{x^2}{4k}\left(\frac{(y+z)(x+y+z)}{x} - (y+z)\right) \\ &= \frac{x(y+z)^2}{4k} = \frac{xa^2}{4k}, \end{aligned}$$

establishing $(*)$. Our proof is complete.

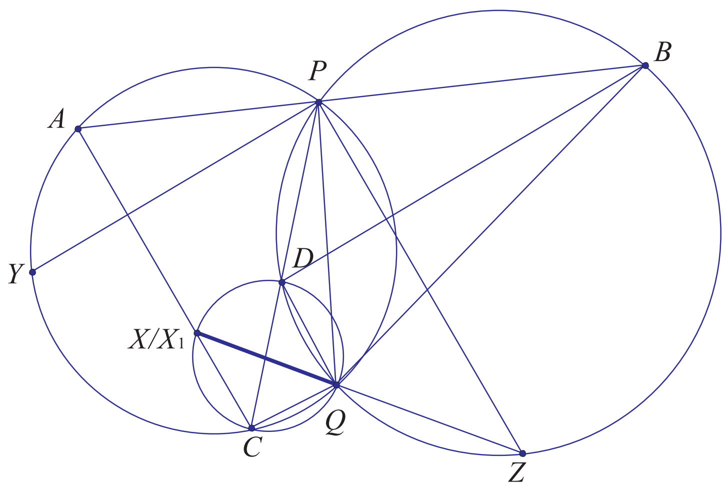
Note: Trigonometric solutions of $(*)$ are also possible.
Query: For a given triangle, how can one construct ω_A and Ω_A by ruler and compass?

(This problem was suggested by Kiran Kedlaya and Sungyoon Kim.)

4 Team Selection Test 2007

1. Circles ω_1 and ω_2 meet at P and Q. Segments AC and BD are chords of ω_1 and ω_2 respectively, such that segment AB and ray CD meet at P. Ray BD and segment AC meet at X. Point Y lies on ω_1 such that $PY \parallel BD$. Point Z lies on ω_2 such that $PZ \parallel AC$. Prove that points Q, X, Y, Z are collinear.

First Solution:

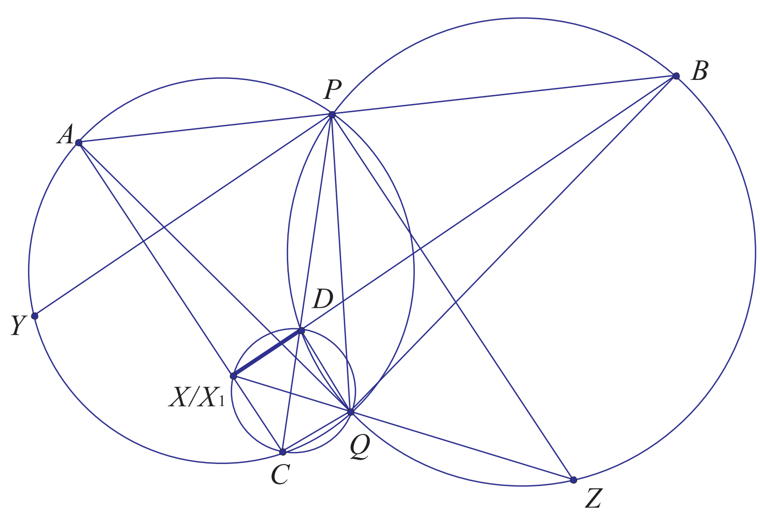


We consider the above configuration. (Our proof can be modified for other configurations.) Let segment AC meet the circumcircle of triangle CQD again (other than C) at X_1.

First, we show that Z, Q, X_1 are collinear. Since $CQDX_1$ is cyclic, $\angle X_1CD = \angle DQX_1$. Since $AC \parallel PZ$, $\angle X_1CD = \angle ACP = \angle CPZ = \angle DPZ$. Since $PDQZ$ is cyclic, $\angle DPZ + \angle DQZ = 180°$. Combining the last three equations, we obtain that

$$\angle DQX_1 + \angle DQZ = \angle X_1CD + \angle DQZ = \angle DPZ + \angle DQZ = 180°;$$

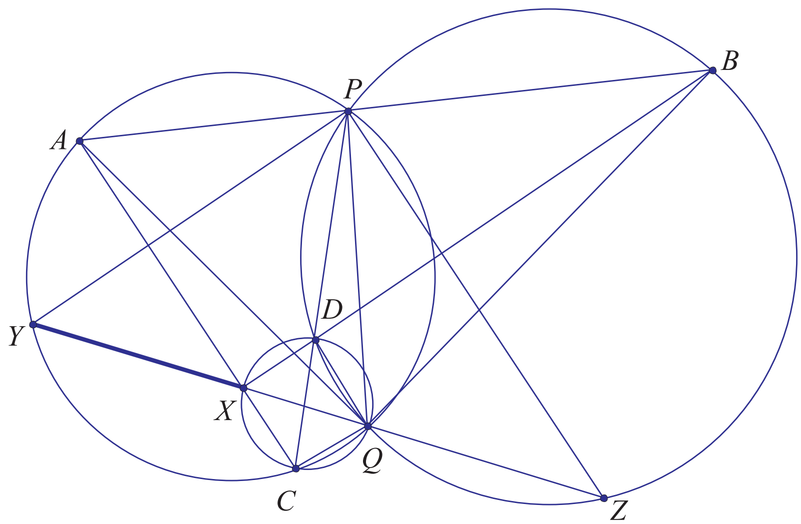
that is, X_1, Q, Z are collinear.



Second, we show that B, D, X_1 are collinear; that is, $X = X_1$. Since $AC \parallel PZ$, $\angle CAP = \angle ZPB$. Since $BPQZ$ is cyclic, $\angle BPZ = \angle BQZ$. It follows that $\angle X_1AB = \angle CAP = \angle BQZ$, implying that $ABQX_1$ is cyclic. Hence $\angle X_1AQ = \angle X_1BQ$. On the other hand, since $BPDQ$ and $APQC$ are cyclic,

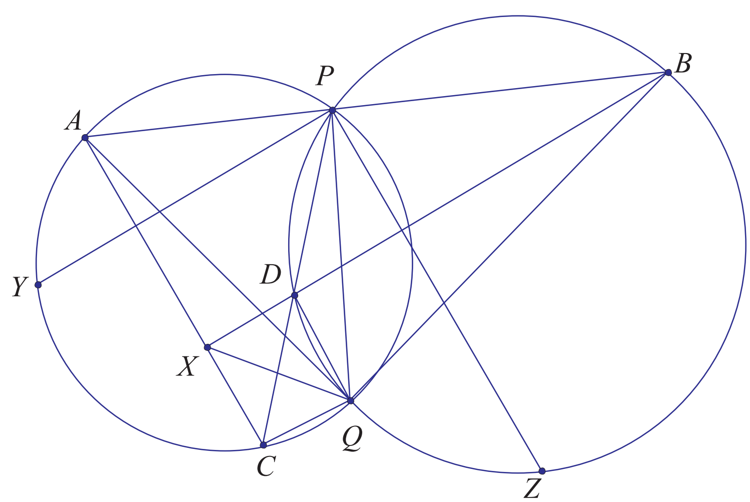
$$\angle QBD = \angle QPD = \angle QPC = \angle QAC = \angle QAX_1.$$

Combining the last two equations, we conclude that $\angle X_1BQ = \angle X_1AQ = \angle QBD$, implying that X_1, D, B are collinear. Since X_1 lies on segment AC, it follows that $X = X_1$. Therefore, we established the fact that Z, Q, X are collinear.



To finish our proof, we show that Y, X, Q are collinear. Since $ABQX$ is cyclic, $\angle BAQ = \angle BXQ$. Since $APQY$ is cyclic, $\angle BAQ = \angle PAQ = \angle PYQ$. Hence $\angle PYQ = \angle BAQ = \angle BXQ$. Since $BX \parallel PY$ and $\angle BXQ = \angle PYQ$, we must have Y, X, Q collinear.

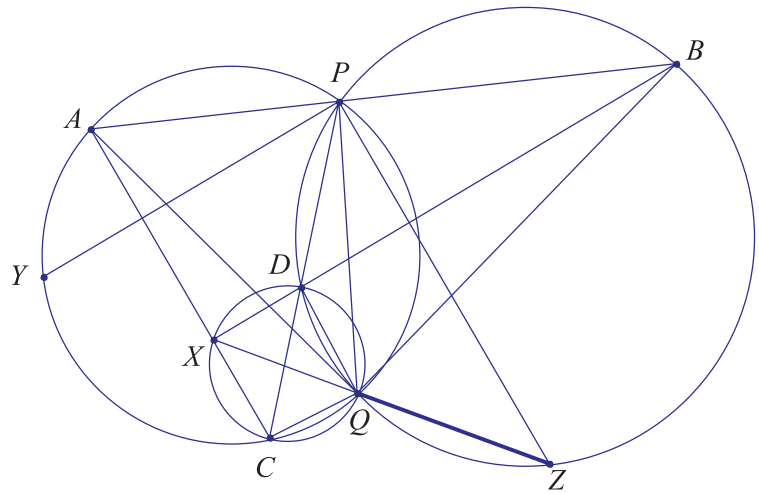
Second Solution:



We claim that $AXQB$ is cyclic. Because $ACQP$ and $PDQB$ are cyclic, we have

$$\angle XAQ = \angle CAQ = \angle CPQ = \angle DPQ = \angle DBQ = \angle XBQ,$$

establishing the claim.



Since $AXQB$ and $BPDQ$ are cyclic, we have

$$\angle QSC = \angle ABQ = \angle PBQ = \angle CDQ,$$

implying that $XDQC$ is cyclic.

Because $XDQC$ is cyclic, $\angle DQX = \angle DCX = \angle PCA$. Since $PZ \parallel AC$, $\angle PCA = \angle CPZ = \angle DPZ$. Hence $\angle DQX = \angle DPZ$. Since $PDQZ$ is cyclic, $\angle DPZ + \angle DQZ = 180°$. Combing the last two equations yields $\angle DQX + \angle DQZ = \angle DPZ + \angle DQZ = 180°$; that is, X, Q, Z are collinear.

Likewise, we can show that Y, X, Q are collinear.

(This problem was suggested by Zuming Feng and Zhonghao Ye.)

2. Let n be a positive integer and let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be two nondecreasing sequences of real numbers such that

$$a_1 + \cdots + a_i \le b_1 + \cdots + b_i \quad \text{for every } i = 1, \ldots, n-1$$

and

$$a_1 + \cdots + a_n = b_1 + \cdots + b_n.$$

Suppose that for any real number m, the number of pairs (i, j) with $a_i - a_j = m$ equals the number of pairs (k, ℓ) with $b_k - b_\ell = m$. Prove that $a_i = b_i$ for $i = 1, \ldots, n$.

Note: It is important to interpret the condition that for any real number m, the number of pairs (i, j) with $a_i - a_j = m$ equals the number of pairs (k, ℓ) with $b_k - b_\ell = m$. It means that we have two identical multi-sets (a multi-set is a set that allows repeated elements)

$$\{a_i - a_j \mid 1 \le i < j \le n\} \quad \text{and} \quad \{b_k - b_\ell \mid 1 \le k < \ell \le n\}.$$

In particular, it gives us that

$$\sum_{1 \le i < j \le n} (a_i - a_j) = \sum_{1 \le k < \ell \le n} (b_k - b_\ell), \tag{$*$}$$

$$\sum_{1\le i<j\le n}(a_i-a_j)^2=\sum_{1\le k<\ell\le n}(b_k-b_\ell)^2 \qquad (**)$$

and

$$\sum_{i,j=1}^{n}|a_i-a_j|=\sum_{i,j=1}^{n}|b_i-b_j|. \qquad (***)$$

We present three solutions. The first solution is based on $(*)$, the second is based on $(**)$, and the third is based on $(***)$.

First Solution: Put $s_n=a_1+\cdots+a_n=b_1+\cdots+b_n$. Then

$$\begin{aligned}2\sum_{i=1}^{n-1}(a_1+\cdots+a_i)&=2(n-1)a_1+2(n-2)a_2+\cdots+2(1)a_{n-1}\\&=(n-1)a_1+(n-3)a_2+\cdots+(1-n)a_n+(n-1)s_n\\&=(n-1)s_n+\sum_{1\le i<j\le n}(a_i-a_j)\end{aligned}$$

and similarly

$$2\sum_{i=1}^{n-1}(b_1+\cdots+b_i)=(n-1)s_n+\sum_{1\le k<\ell\le n}(b_k-b_\ell).$$

By $(*)$, these two quantities are equal, so

$$2\sum_{i=1}^{n-1}(a_1+\cdots+a_i)=2\sum_{i=1}^{n-1}(b_1+\cdots+b_i).$$

Consequently, each of the inequalities $a_1+\cdots+a_i\le b_1+\cdots+b_i$ for $i=1,\ldots,n-1$ must be an equality. Since we also have equality for $i=n$ by assumption, we deduce that $a_i=b_i$ for $i=1,\ldots,n$, as desired.

Second Solution: Expanding both sides of $(**)$ yields

$$(n-1)\sum_{i=1}^{n}a_i^2+2\sum_{1\le i<j\le n}a_ia_j=(n-1)\sum_{i=1}^{n}b_i^2+2\sum_{1\le k<\ell\le n}b_kb_\ell.$$

Squaring both sides of the given equation $a_1+\cdots+a_n=b_1+\cdots+a_n$ gives

$$\sum_{i=1}^{n}a_i^2+2\sum_{1\le i<j\le n}a_ia_j=\sum_{i=1}^{n}b_i^2+2\sum_{1\le k<\ell\le n}b_kb_\ell.$$

From the above relations we easily deduce that

$$\sum_{i=1}^{n}a_i^2=\sum_{i=1}^{n}b_i^2.$$

By the Cauchy-Schwartz inequality, we obtain that

$$\left(\sum_{i=1}^{n}b_i^2\right)^2=\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)\ge\left(\sum_{i=1}^{n}a_ib_i\right)^2$$

or

$$\sum_{i=1}^{n} b_i^2 \geq \left|\sum_{i=1}^{n} a_i b_i\right| \geq \sum_{i=1}^{n} a_i b_i. \tag{†}$$

We set $s_i = a_1 + \cdots + a_i$ and $t_i = b_1 + \cdots + b_i$ for every $1 \leq i \leq n$. By Abel's summation formula, we have

$$\begin{aligned}\sum_{i=1}^{n} a_i b_i &= s_1 b_1 + [s_2 - s_1]b_2 + [s_3 - s_2]b_3 + \cdots + [s_n - s_{n-1}]b_n \\ &= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_n b_n.\end{aligned}$$

By the given conditions, $s_i \leq t_i$ and $b_i - b_{i+1} \leq 0$ for every $1 \leq i \leq n-1$ and $s_n = t_n$. It follows that

$$\begin{aligned}\sum_{i=1}^{n} a_i b_i &\geq t_1(b_1 - b_2) + t_2(b_2 - b_3) + \cdots + t_{n-1}(b_{n-1} - b_n) + t_n b_n \\ &= t_1 b_1 + [t_2 - t_1]b_2 + [t_3 - t_2]b_3 + \cdots + [t_n - t_{n-1}]b_n = \sum_{i=1}^{n} b_i^2.\end{aligned}$$

Combining the last inequality and (†), we conclude that the equality case holds for every inequality we discussed above. In particular, $s_i = t_i$ for $i = 1, \ldots, n$. These inequalities immediately give us $a_n = b_n$, $a_{n-1} = b_{n-1}$, $\ldots$, $a_1 = b_1$ and the problem is solved.

Third Solution: If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are two nonincreasing sequences, we say that u *majorizes* v if $u_1 + \cdots + u_n = v_1 + \cdots + v_n$ and $u_1 + \cdots + u_i \geq v_1 + \cdots + v_i$ for $i = 1, 2, \ldots, n-1$. It is not difficult to see that $(a_n, \cdots, a_1)$ majorizes $(b_n, \ldots, b_1)$. By a theorem of Birkhoff, it follows that there are constants $c_\sigma \in (0, 1]$, where σ runs over some set S of permutations of $\{1, \ldots, n\}$, with $\sum_{\sigma \in S} c_\sigma = 1$ and

$$\sum_{\sigma \in S} c_\sigma a_{\sigma(i)} = b_i \quad \text{for } i = 1, 2, \ldots, n.$$

We will prove the inequality

$$\sum_{i,j=1}^{n} |a_i - a_j| \geq \sum_{i,j=1}^{n} |b_i - b_j|,$$

with equality if and only if $a_i = b_i$ for $i = 1, \ldots, n$. With this result, we complete our proof by noting $(***)$.

We have

$$\begin{aligned}\sum_{i,j=1}^{n} |a_i - a_j| &= \sum_{\sigma \in S} c_\sigma \sum_{i,j=1}^{n} |a_i - a_j| = \sum_{\sigma \in S} c_\sigma \sum_{i,j=1}^{n} |a_{\sigma(i)} - a_{\sigma(j)}| \\ &= \sum_{i,j=1}^{n} \sum_{\sigma \in S} c_\sigma |a_{\sigma(i)} - a_{\sigma(j)}| \geq \sum_{i,j=1}^{n} \left|\sum_{\sigma \in S} c_\sigma (a_{\sigma(i)} - a_{\sigma(j)})\right| = \sum_{i,j=1}^{n} |b_i - b_j|,\end{aligned}$$

using the fact that $|x_1| + \cdots + |x_m| \geq |x_1 + \cdots + x_m|$ for all real numbers $x_1, \ldots, x_m$.

This establishes the desired inequality; it remains to check the equality condition. For this, we must have

$$\sum_{\sigma \in S} c_\sigma |a_{\sigma(i)} - a_{\sigma(j)}| = \left|\sum_{\sigma \in S} c_\sigma (a_{\sigma(i)} - a_{\sigma(j)})\right|$$

for each pair i and j; in particular, for each pair i and j, the sign of $a_{\sigma(i)} - a_{\sigma(j)}$ must be the same for all $\sigma \in S$ for which $a_{\sigma(i)} \neq a_{\sigma(j)}$. It follows by the lemma below that the sequence $a_{\sigma(1)}, \dots, a_{\sigma(n)}$ itself must be the same for all $\sigma \in S$, yielding $a_i = b_i$.

Lemma *Let σ be a permutation of $\{1, \dots, n\}$, and let $(a_1, \dots, a_n)$ be an n-tuple of real numbers. If*

$$(a_{\sigma(1)}, \dots, a_{\sigma(n)}) \neq (a_1, \dots, a_n),$$

then there exist $i, j \in \{1, \dots, n\}$ such that $a_i < a_j$ but $a_{\sigma(i)} > a_{\sigma(j)}$.

Proof: We proceed by induction on n. There is no harm in assuming that $a_1 \leq \cdots \leq a_n$. Let m be the least integer for which $a_m = a_n$. If $\{m, \dots, n\} = \{\sigma(m), \dots, \sigma(n)\}$, then σ also permutes $\{1, \dots, m-1\}$ and we can reduce to that case. Otherwise, there is some $i \geq m$ such that $\sigma(i) < m$, and there is some $j < m$ such that $\sigma(j) \geq m$. This pair i, j has the desired property. ■

Note: The problem can be made slightly simpler by requiring $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$, as this avoids the lemma at the end of the third solution. This solution also reveals the relation between this problem and Muirhead's inequality. For more details of majorization, Muirhaed's inequality, and Birkhoff's theorem, one may visit the site `en.wikipedia.org/wiki/Muirhead's_inequality`. Note also that it is an easy exercise with generating functions to construct counterexamples if you drop the majorization condition, even if one ignores cases where the two sets differ by a translation plus a reflection.

(This problem was suggested by Kiran Kedlaya.)

3. Let θ be an angle in the interval $(0, \pi/2)$. Given that $\cos\theta$ is irrational, and that $\cos k\theta$ and $\cos[(k+1)\theta]$ are both rational for some positive integer k, show that $\theta = \pi/6$.

Note: We present two solutions. Both solutions are based on the following facts.

Lemma 1 *For every positive integer n, there is a monic polynomial (that is, a polynomial with leading coefficient 1) $S_n(x)$ with integer coefficients such that $S_n(2\cos\alpha) = 2\cos n\alpha$.*

Proof: We induct on n. The base cases $n = 1$ and $n = 2$ are trivial by taking $S_1(x) = x$ and $S_2(x) = x^2 - 2$. Assume the statement is true for $n \leq m$. Note that by the addition-to-product formulas, $2\cos[(m+1)\alpha] + 2\cos[(m-1)\alpha] = 4\cos m\alpha \cos\alpha$. Thus $S_{m+1}(x) = xS_m(x) - S_{m-1}(x)$ satisfies the conditions of the problem, completing the induction. ■

Lemma 2 *If $\cos\alpha$ is rational and $\alpha = r\pi$ for some rational number r, then the possible values of $\cos\alpha$ are $0, \pm 1, \pm\frac{1}{2}$.*

Proof: Since r is rational, there exists positive integer n such that rn is an even integer. By lemma 1, $S_n(2\cos\alpha) = 2\cos(n\alpha) = 2\cos(rn\pi) = 1$; that is, $2\cos\alpha$ is a ration root of the monic polynomial $S_n(x)$ with integer coefficients. By Gauss' lemma, $2\cos\alpha$ must take integer values. Since $-1 \leq \cos\alpha \leq 1$, the possible values of $2\cos\alpha$ are $0, \pm 1, \pm 2$ ■

First Solution: We note that if $\cos x$ is rational, then $\cos nx$ is rational for every positive integer n. Indeed, this fact follows from a easy induction on n by noting the product-to-sum formula

$$2\cos n\theta \cos\theta = \cos[(n+1)\theta] + \cos[(n-1)\theta].$$

Thus both $\cos(k^2\theta) = \cos[k(k\theta)]$ and $\cos[(k^2-1)\theta] = \cos[(k-1)(k+1)\theta]$ are rational. By the Addition and subtraction formulas, we have

$$\cos[(k+1)\theta] = \cos k\theta\cos\theta - \sin k\theta\sin\theta \quad \text{and} \quad \cos(k^2\theta) = \cos[(k^2-1)\theta]\cos\theta - \sin[(k^2-1)\theta]\sin\theta.$$

Setting $r_1 = \cos k\theta$, $r_2 = \cos[(k+1)\theta]$, $r_3 = \cos[(k^2-1)\theta]$, $r_4 = \cos(k^2\theta)$, , and $x = \cos\theta$ in the above equations yields

$$r_2 = r_1 x \pm \sqrt{(1-r_1^2)(1-x^2)} \quad \text{and} \quad r_4 = r_3 x \pm \sqrt{(1-r_3^2)(1-x^2)},$$

or

$$\pm\sqrt{(1-r_1^2)(1-x^2)} = r_2 - r_1 x \quad \text{and} \quad \pm\sqrt{(1-r_3^2)(1-x^2)} = r_4 - r_3 x.$$

Squaring these two equations and subtracting the resulting equations gives

$$2(r_1r_2 - r_3r_4)x = r_1^2 + r_2^2 - (r_3^2 + r_4^2).$$

Since r_1, r_2, r_3, r_4 are rational and x is irrational, we must have $r_1r_2 - r_3r_4 = 0$ or

$$\cos k\theta\cos[(k+1)\theta] = \cos(k^2\theta)\cos[(k^2-1)\theta].$$

By the product-to-sum formulas, we derive

$$\frac{\cos[(2k+1)\theta] - \cos\theta}{2} = \frac{\cos[(2k^2-1)\theta] - \cos\theta}{2}$$

or $\cos[(2k+1)\theta] - \cos[(2k^2-1)\theta] = 0$. By the sum-to-product formulas, we obtain

$$2\sin[(k-k^2+1)\theta]\sin[(k^2+k)\theta] = 0,$$

implying that either $(k-k^2+1)\theta$ or $(k^2+k)\theta$ is a integral multiple of π. Since k is an integer, we conclude that $\theta = r\pi$ for some rational number r.

Considering lemma 2 for $\alpha = k\theta$ and $\alpha = (k+1)\theta$, the possible values of $\cos k\theta$ and $\cos[(k+1)\theta]$ are $0, \pm 1, \pm\frac{1}{2}$. Consequently, both $k\theta$ and $(k+1)\theta$ is a integral multiple of $\frac{\pi}{6}$. Since $0 < \theta = k\theta - (k-1)\theta < \frac{\pi}{2}$, the only possible values of θ are $\frac{\pi}{3}$ and $\frac{\pi}{6}$. Since $\cos\theta$ is irrational, $\theta = \frac{\pi}{6}$.

Second Solution: (Based on the work by Kiran Kedlaya) We maintain the notations used in the first proof. Then $s = 2\cos\theta$ is a root of $S_k(x) - 2r_1$ and $S_{k+1}(x) - 2r_2$ by the definition of S_n. Define

$$Q(x) = \gcd(S_k(x) - 2r_1, S_{k+1}(x) - 2r_2)$$

where the gcd is taken over the field of rational numbers. Then $Q(x)$ is a polynomial with rational coefficients, so the sum of its roots (with multiplicities) is rational. Since s is assumed not to be rational, there must be at least one other distinct root t of $Q(x)$.

Note that the k distinct reals $2\cos(\theta + 2\pi a/k)$ for $a = 0, 1, \ldots, k-1$ form k roots of the degree k polynomial $S_k(x) - 2r_1$, so they compose all of its roots. Similarly, all of the roots of $S_{k+1}(x) - 2r_2$ have the form $2\cos(\theta + 2\pi b/(k+1))$ for $b = 0, 1, \ldots, k$. Note that s and t are roots of $Q(x)$. Therefore roots of both $S_k(x) - 2r_1$ and $S_k(x) - 2r_2$, and so they must have at least two distinct common roots. Each root r of $Q(x)$ must thus satisfy

$$r = 2\cos(\theta + 2\pi a/k) = 2\cos(\theta + 2\pi b/(k+1))$$

for some a and b. We either have $\theta + 2\pi a/k = \theta + 2\pi b/(k+1)$ and thus $r = 2\cos\theta$ or $\theta + 2\pi a/k = -\theta - 2\pi b/(k+1)$ and thus

$$\theta = -\frac{\pi[(a+b)k + a]}{k(k+1)}.$$

In the first case, we obtain s, so t must lead to the second value of θ, as $s \neq t$.

Therefore, we can write $\theta = \frac{\pi c}{k(k+1)}$ for some integer c. By Lemma 2, c/k and $c/(k+1)$ must both be multiples of $1/6$, since $\cos k\theta = \cos \frac{c\pi}{k+1}$ and $\cos(k+1)\theta = \cos \frac{c\pi}{k}$ are rational. Therefore, $\theta = \frac{c\pi}{k} - \frac{c\pi}{k+1}$ is a multiple of $\pi/6$. Since t is not rational, θ can only be $\pi/6$.

(This problem was suggested by Zhigang Feng, Zuming Feng, and Weigu Li.)

4. Determine whether or not there exist positive integers a and b such that a does not divide $b^n - n$ for all positive integers n.

Note: The answer is *no*. We present two solutions, based on the following fact.

Lemma 1. *Given positive integers a and b, for sufficiently large n we have that*

$$b^{n+\varphi(a)} \equiv b^n \pmod{a}.$$

(The function φ is the Euler's totient function: For any positive integer m we denote by $\varphi(m)$ the number of all positive integers n less than m that are relatively prime to m.)

Proof: Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are distinct primes. We know that φ is a multiplicative functions, that is,

$$\varphi(a) = \varphi\left(p_1^{\alpha_1}\right)\varphi\left(p_2^{\alpha_2}\right)\cdots\varphi\left(p_k^{\alpha_k}\right) = \left(p_1^{\alpha_1} - p_1^{\alpha_1 - 1}\right)\cdots\left(p_k^{\alpha_k} - p_1^{\alpha_k - 1}\right) = a\left(1 - \frac{1}{p_1}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

In particular, $\varphi(p_i^{\alpha_i}) \mid \varphi(a)$ for each $1 \le i \le k$ and $\varphi(a) < a$.

For each p_i, if p_i divides b, then $b^n \equiv 0 \pmod{p_i^{\alpha_i}}$ for $n \ge \alpha_i + 1$. Hence $b^{n+\varphi(a)} \equiv b^n b^{\varphi(a)} \equiv b^n \equiv 0 \pmod{p_i^{\alpha_i}}$ for $n \ge \alpha_i + 1$; if p_i does not divide b, then $\gcd(p_i^{\alpha_i}, b) = 1$. By Euler's theorem, we have $b^{\varphi(p_i^{\alpha_i})} \equiv 1 \pmod{p_i^{\alpha_i}}$. Since $\varphi(p_i^{\alpha_i}) \mid \varphi(a)$, we have $b^{n+\varphi(a)} \equiv b^n \pmod{p_i^{\alpha_i}}$. Therefore, for each p_i, we have some n_i such that for all $n > n_i$, $b^{n+\varphi(a)} \equiv b^n \pmod{p_i^{\alpha_i}}$. Thus, take $N = \max\{n_i\}$ and note that for all $n > N$, we have $b^{n+\varphi(a)} \equiv b^n \pmod{p_i^{\alpha_i}}$ for all $i \le i \le k$. Since p_i are distinct, $b^{n+\varphi(a)} \equiv b^n \pmod{a}$, as desired. ■

First Solution: For any positive integers a and b, we claim that there exists infinitely many n such that a divides $b^n - n$.

We establish our claim by strong induction on a. The base case of $a = 1$ holds trivially. Now, suppose that the claim holds for all $a < a_0$. Since $\varphi(a) < a$, by the induction hypothesis and by lemma 1, there are infinitely many n such that

$$\varphi(a) \mid (b^n - n) \quad \text{and} \quad b^{n+\varphi(a)} \equiv b^n \pmod{a}.$$

For each of such n, set

$$t = \frac{b^n - n}{\varphi(a)} \quad \text{and} \quad n_1 = b^n = n + t\varphi(a).$$

It follows that

$$b^{n_1} - n_1 \equiv b^{n+t\varphi(a)} - (n + t\varphi(a)) \equiv b^n - n - t\varphi(a) \equiv 0 \pmod{a}.$$

Then, we see that n_1 satisfies the desired property. By the induction hypothesis, there are clearly infinitely many $n_1 = b^n$ satisfies the conditions of the claim for a, completing the induction.

Second Solution: We prove no such a, b exist by proving the following: for any a, b, there is an arithmetic progression $n \equiv h \pmod{m}$, with m divisible only by primes less than or equal to the greatest prime factor of a, such that $b^n \equiv n \pmod{a}$ for all sufficiently large n satisfying $n \equiv h \pmod{m}$.

Let us induct on highest prime divisor of a. The result is trivial for $a = 1$. Let p be a prime, and suppose that the result is true whenever all the prime divisors of a are less than p. Now, suppose that p is the greatest prime divisor of some a, and write $a = p^e a_1$, where a_1 has all prime factors less than p. By the induction hypothesis, there is an arithmetic progression $n \equiv h_1 \pmod{m_1}$, with m_1 divisible only by primes strictly less than p, such that for $n \equiv h_1 \pmod{m_1}$ sufficiently large, $b^n \equiv n \pmod{a_1}$. There is no harm in assuming that $p - 1$ divides m_1. In this case, in this arithmetic progression, b^n is eventually constant modulo p due to the lemma. We can thus choose a congruence modulo p so that for n an appropriate residue class modulo $m_1 p$, $b^n \equiv n \pmod{p}$. In this progression, b^n is constant modulo p^2, so we can refine our choice of n modulo $m_1 p$ to a choice of n modulo $m_1 p^2$ to force $b^n \equiv n \pmod{p^2}$. We can then repeat the above process until we obtain $b^n \equiv n \pmod{p^e}$. Since we originally had $b^n \equiv n \pmod{a_1}$, combining the two congruences using the Chinese Remainder Theorem gives us $b^n \equiv n \pmod{a}$ for all sufficiently large n in congruence class generated at the last step. This completes the induction.

Note: The key idea in both solutions is to reduce a, and the two solutions differ by how *fast* the reduction takes place. While the second solution removes the prime divisors of a one by one starting from the greatest, the first solution reduces a to $\phi(a)$.

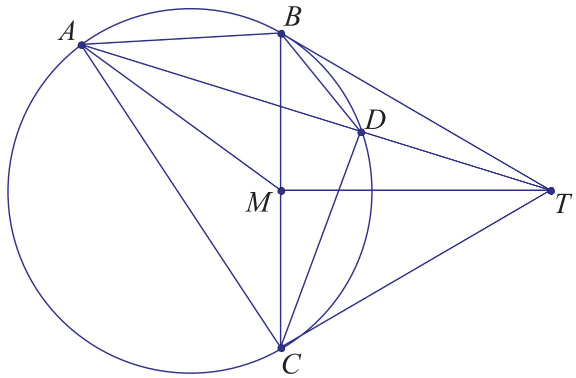
These solutions remind us problem 3 of USAMO 1991:

> Show that, for any fixed integer $n \geq 1$, the sequence
>
> $$2,\ 2^2,\ 2^{2^2},\ 2^{2^{2^2}}, \ldots \pmod{n}$$
>
> is eventually constant. (The tower of exponents is defined by $a_1 = 2,\ a_{i+1} = 2^{a_i}$.)

(This problem was suggested by Thomas Mildorf.)

5. Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lies on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

First Solution: (Based on the work by Oleg Golberg) We start with a important geometry observation.



Lemma *Triangle ABC inscribed in circle ω Lines BT and CT are tangent to ω. Let M be the midpoint of side BC. Then $\angle BAT = \angle CAM$. (Line AT is a symmedian of triangle.)*

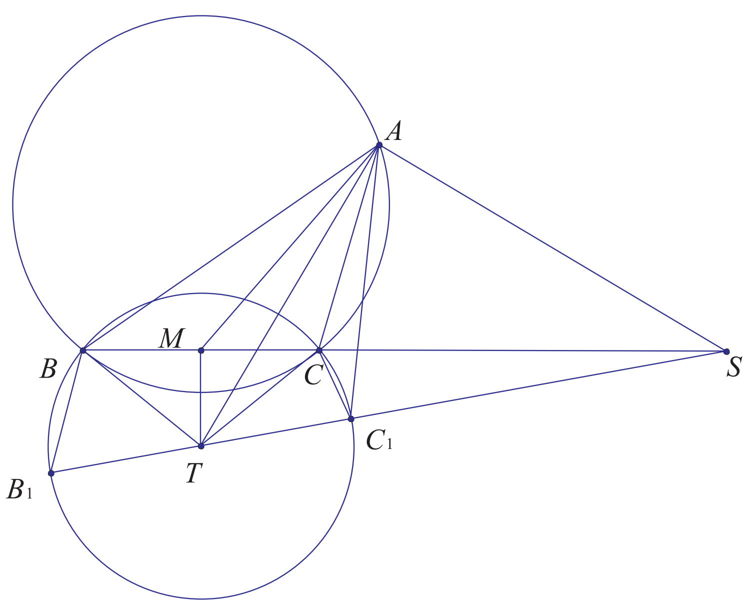
Proof: (We consider the above configuration. If $\angle BAC$ is obtuse, our proof can be modified slightly.) Let D denote the second intersection (other than A) of line AT and circle ω. Because BT is tangent to ω at B, $\angle TBD = \angle TAB$. Hence triangles TBD and TAB are similar, implying that $BD/AB = TB/TA$. Likewise, triangles TCD are TAC are similar and $CD/AC = TC/TA$. By equal tangents, $TB = TC$. Consequently, we have $BD/AB = TB/TA = TC/TA = CD/AC$, implying that

$$BD \cdot AC = CD \cdot AB.$$

By the **Ptolemy's theorem** to cyclic quadrilateral $ABDC$, we have $BD \cdot AC + AB \cdot CD = AD \cdot BC$. Combining the last two equations, we obtain that $2BD \cdot AC = AD \cdot BC$ or

$$\frac{AC}{AD} = \frac{BC}{2BD} = \frac{MC}{BD}.$$

Further considering that $\angle ACM = \angle ACB = \angle ADB$ (since $ABDC$ is cyclic), we conclude that triangle ABD is similar to triangle AMC, implying that $\angle BAT = \angle BAD = \angle CAM$. ■



Because BT is tangent to ω, $\angle CBT = \angle CAB$, and so

$$\angle TBA = \angle ABC + \angle CBT = \angle ABC + \angle CAB = 180^\circ - \angle BCA.$$

By the lemma, we have $\angle BAT = \angle CAM$. Applying the Law of Sines to triangles BAT and CAM, we obtain

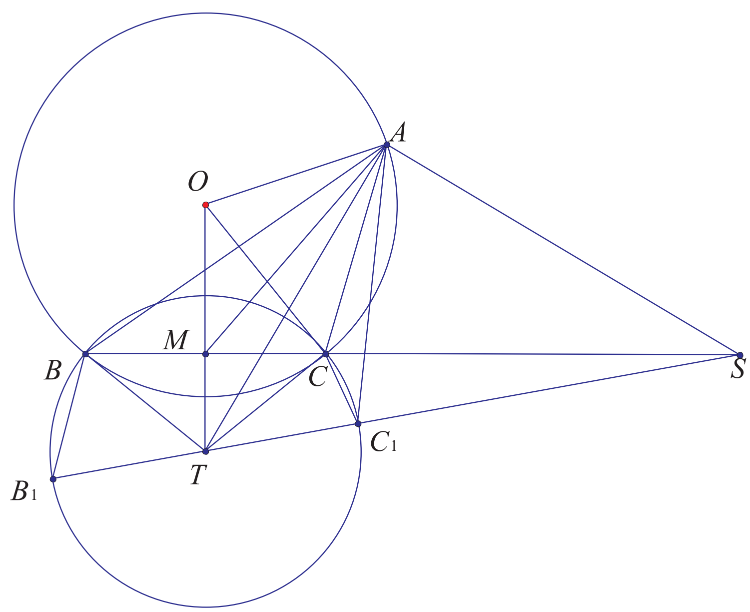
$$\frac{BT}{AT} = \frac{\sin\angle BAT}{\sin\angle TBA} = \frac{\sin\angle CAM}{\sin\angle BCA} = \frac{MC}{AM}.$$

Note that $TB = TC_1$. Thus, $TC_1/TA = MC/MA$. By equal tangents, $TB = TC$. In isosceles triangle BTC, M is the midpoint of base BC. Consequently, $\angle TMS = \angle TAC = \angle TAS = 90^\circ$, implying that $TMAP$ is cyclic. Hence $\angle AMC = \angle ATC_1$. Because

$$\frac{AM}{AT} = \frac{MC}{TC_1} \qquad (*)$$

and $\angle AMC = \angle ATC_1$, triangles MAC and TAC_1 are similar. Because $BC/BM = B_1C_1/TC_1 = 2$, triangles ABC and AB_1C_1 are similar.

Second Solution: (By Alex Zhai) We maintain the notations in the first proof. As shown at the end of the first proof, it suffices to show that $(*)$.

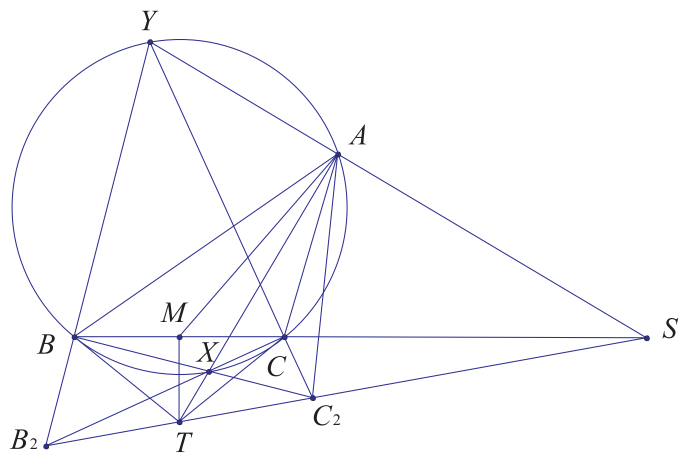


Let O be the circumcenter of ABC. Note that triangles OMC, OCT are similar to each other, implying that $OM/OC = OC/OT$ or $OM \cdot OT = OC^2 = OA^2$. Thus triangles OAM and OTA are also similar to each other. Further note that triangles OMC and CMT are also similar to each other. These similarities (which amount to the circumcircle of ABC being a circle of Apollonius) give

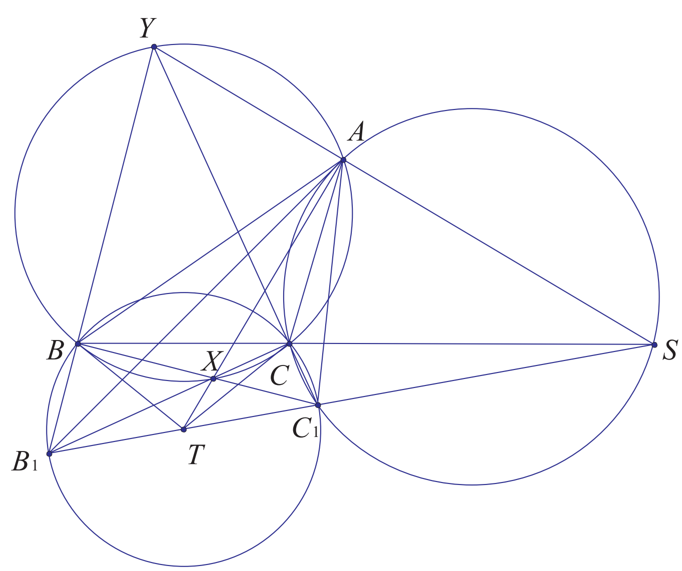
$$\frac{AM}{AT} = \frac{OM}{OA} = \frac{OM}{OC} = \frac{MC}{CT} = \frac{MC}{TB} = \frac{MC}{TC_1},$$

which is $(*)$.

Third Solution: (Based on work by Sherry Gong) We maintain the notations of the previous solutions. Let ω intersect lines AT and AS again at X and Y (other than A), respectively. Let lines YB and CX meet at B_2, and let YC and BX meet at C_2. Applying the Pascal's theorem to cyclic (degenerated) hexagon $BBYAXC$ shows that intersections of three pairs of lines BB and AX, BY and XC, and YA and CB are collinear; that is, B_2, C_2, S are collinear. Likewise, applying the Pascal's theorem to cyclic (degenerated) hexagon $CCYAXB$ shows that B_2, C_2, T are collinear. We conclude that B_2, C_2, S, T are collinear.



Since $ACXY$ is cyclic, $\angle YCX = \angle YAX = 180^\circ - \angle XAB = 90^\circ$. Thus $\angle B_2CC_2 = \angle XCC_2 = 180^\circ - \angle YCX = 90^\circ$. Likewise, $\angle C_2BB_2 = 90^\circ$. It follows that BCC_2B_2 is inscribed in a circle with B_2C_2 as its diameter. Thus the circumcenter of this circle is the intersection of lines ST and the perpendicular of segment BC. This circumcenter must thus be T, and consequently, $B_2 = B_1$ and $C_2 = C_1$.



Because $AYBC$ and B_1C_1CB are cyclic, by Miquel's theorem, $SACC_1$. (Indeed, $\angle ACB = 180^\circ - \angle B_1YS$ and $\angle BCC_1 = \angle 180^\circ - \angle YB_1S$ lead to $\angle ACC_1 = 360^\circ - \angle ACB - \angle BCC_1 = 180^\circ - YSB_1$.)

Also, by Miquel's theorem, YAC_1B_1 is cyclic. (Indeed, $\angle C_1AX = \angle C_1CS = \angle SB_1Y$.) By these cyclic quadrilaterals, it is not difficult to obtain $\angle ACS = \angle AC_1S$ (or $\angle ACB = \angle AC_1B_1$) and $\angle ABC = \angle AYC = \angle AYC_1 = \angle AB_1C_1$. Consequently, triangles ABC and AB_1C_1 are similar to each other.

Note: The last approach reveals the problem posers' motivation. We can view the BCC_1B_1-$\{S,T\}$ as a complete quadrilateral. Then A is is its Miquel's point. This problem combines two properties of complete quadrilateral and its Miquel's points: (1) A lies on ST if and only if BCC_1B_1 is cyclic; (2) the line through A perpendicular to ST passes through the circumcenter of B_1BC1.

(This problem was suggested by Zuming Feng and Zhonghao Ye.)

6. For a polynomial $P(x)$ with integer coefficients, $r(2i-1)$ (for $i = 1, 2, 3, \ldots, 512$) is the remainder obtained when $P(2i-1)$ is divided by 1024. The sequence

$$(r(1), r(3), \ldots, r(1023))$$

is called the *remainder sequence* of $P(x)$. A remainder sequence is called *complete* if it is a permutation of $(1, 3, 5, \ldots, 1023)$. Prove that there are no more than 2^{35} different complete remainder sequences.

Solution: Define the polynomials

$$\begin{aligned}
Q_0(x) &= b_0,\\
Q_1(x) &= b_1\ (x+1),\\
Q_2(x) &= b_2\ (x+1)(x+3),\\
Q_3(x) &= b_3\ (x+1)(x+3)(x+5),\\
Q_4(x) &= b_4\ (x+1)(x+3)(x+5)(x+7),\\
Q_5(x) &= b_5\ (x+1)(x+3)(x+5)(x+7)(x+9),\\
Q_6(x) &= b_6\ (x+1)(x+3)(x+5)(x+7)(x+9)(x+11),
\end{aligned}$$

where

$$b_0 = 2^{10}, \quad b_1 = 2^9, \quad , b_2 = 2^7 \quad , b_3 = 2^6 \quad b_4 = 2^3, \quad b_5 = 2^2, \quad b_6 = 2^0.$$

The product of i consecutive even integers is divisible by $2^i \cdot i!$. Therefore, for $i = 0, 1, 2, 3, 4, 5, 6$, we obtain that the product of i consecutive even integers is divisible by $2^0, 2^1, 2^3, 2^4, 2^7, 2^8, 2^{10}$, respectively. This implies that, for any odd integer x and $i = 0, \ldots, 6$, $Q_i(x)$ is divisible by 2^{10}.

A polynomial $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ with integer coefficients is called *reduced* if, for $i = 0, \ldots, 5$,

$$0 \le a_i < b_i. \tag{$\dagger$}$$

Clearly, there are exactly $b_0b_1 \ldots b_5 = 2^{10+9+7+6+3+2} = 2^{37}$ distinct reduced polynomials.

We show that, for every polynomial $P(x)$ with integer coefficients, there exists a reduced polynomial $\bar{P}(x)$ such that $P(x)$ and $\bar{P}(x)$ have the same remainder sequence.

First note that, for $i = 0, \ldots, 6$, and any polynomial $R(x)$ with integer coefficients $P(x)$ and $P(x) - R(x)Q_i(x)$ have the same remainder sequence. This follows from the fact that $Q_i(x)$ is divisible by 2^{10}, for any odd integer x.

If the degree d of $P(x) = a_0 + \cdots + a_d x^d$ is higher than 5 we may replace $P(x)$ by $P(x) - a_d x^{d-6} Q_6(x)$. Indeed, the polynomial $P(x) - a_d x^{d-6} Q_6(x)$ has smaller degree than $P(x)$ and has the same remainder sequence as $P(x)$. We may continue this until we obtain a polynomial that of degree at most 5 that has the same remainder sequence as $P(x)$.

We assume now that $P(x)$ has degree no higher than 5. If $P(x)$ is reduced we are done. Otherwise, let i be the highest degree of a coefficient a_i of x^i that does not satisfy the range condition (†) If q is the quotient obtained by dividing a_i by b_i then $P(x)$ and $P(x) - qQ_i(x)$ have the same remainder sequence and the coefficient at degree i in $P(x) - qQ_i(x)$ is in the correct range $0, \dots, b_i - 1$.

We repeat this procedure with the next highest degree that has a coefficient out of range until we reach a reduced polynomial that has the same remainder sequence as $P(x)$.

We now consider the 2^{37} reduced polynomials.

Let $a = 2^9 + 1$ and $b = 1$. Then $P(a) - P(b) = (a-b)(a_1 + a_2 A_2 + a_3 A_3 + a_4 A_4 + a_5 A_5)$, where $A_2 = a + b$, $A_3 = a^2 + ab + b^2$, $A_4 = a^3 + a^2 b + ab^2 + b^3$ and $A_5 = a^4 + a^3 b + a^2 b^2 + ab^3 + b^4$. Since both a and b are odd, A_2 and A_4 are even, A_3 and A_d are odd, and the parity of $a_1 + a_2 A_2 + a_3 A_3 + a_4 A_4 + a_5 A_5$ is the same as the parity of $a_1 + a_3 + a_5$. Therefore, if $a_1 + a_3 + a_5$ is even $P(a) - P(b)$ is divisible by 2^{10} and the sequence of remainders of $P(x)$ is not a permutation.

For an odd integer x, the parity of $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$ is the same as the parity of the sum $a_0 + a_1 + \cdots + a_5$. Thus, only polynomials with odd sum of coefficients have odd remainders.

Therefore, there are no more remainder sequences that are permutations of $1, 3, \dots, 1023$ than there are reduced polynomials $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$ for which both $a_1 + a_3 + a_5$ and $a_0 + a_1 + a_2 + a_3 + a_4 + a_5$ are odd.

There are exactly 2^{36} reduced polynomials for which $a_1 + a_3 + a_5$ is odd. This can be seen by pairing up every reduced polynomial $P(x)$ in which a_1 is even with the polynomial $P(x) + x$. Exactly one of the two polynomials in each such pair has odd sum $a_1 + a_3 + a_5$.

There are exactly 2^{35} reduced polynomials for which both $a_1 + a_3 + a_5$ and $a_0 + a_1 + a_2 + a_3 + a_4 + a_5$ are odd. This can be seen by pairing up every reduced polynomial $P(x)$ in which $a_1 + a_3 + a_5$ is odd and a_0 is even with the polynomial $P(x) + 1$. Exactly one of the two polynomials in each such pair has odd sum $a_0 + a_1 + a_2 + a_3 + a_4 + a_5$.

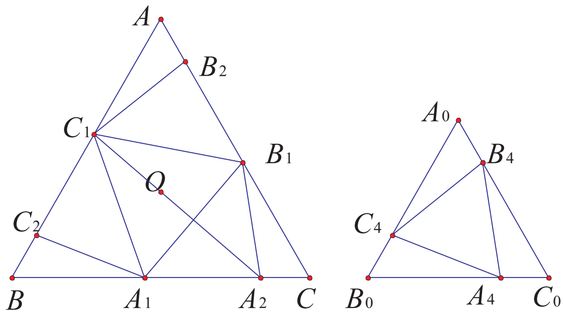
Note: It can be proved that there are exactly 2^{35} different remainder sequences that are permutations of $1, 3, \dots, 1023$.

(This problem was suggested by Danilo Gligoroski, Smile Markovski, and Zoran Šunić.)

5 IMO 2005

1. Six points are chosen on the sides of an equilateral triangle ABC: A_1 and A_2 on BC, B_1 and B_2 on CA, and C_1 and C_2 on AB. These points are vertices of a convex equilateral hexagon $A_1A_2B_1B_2C_1C_2$. Prove that lines A_1B_2, B_1C_2, and C_1A_2 are concurrent.

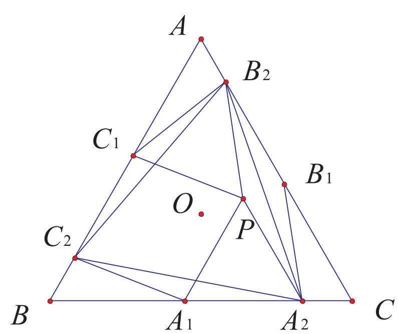
First Solution: (Based on work by Hansheng Diao from China) Set $x = AB$ and $s = A_1A_2$. We construct an equilateral triangle $A_0B_0C_0$ with $A_0B_0 = x - s$. Points C_4, A_4, and B_4 lie on sides A_0B_0, B_0C_0, and C_0A_0, respectively, satisfying $C_4B_0 = C_2B$, $A_4C_0 = A_2C$, and $B_4A_0 = B_2A$. Then it is easy to obtain that $B_0A_4 = BA_1$, $C_0B_4 = CB_1$, and $A_0C_4 = AC_1$. We obtain three pairs of congruent triangles, namely, AB_2C_1 and $A_0B_4C_4$, BC_2A_1 and $B_0C_4A_4$, and CA_2B_1 and $C_0A_4B_4$. (Indeed, we are sliding the three *corner* triangles together.)



It follows that $A_4B_4 = B_4C_4 = C_4A_4 = s$; that is, triangle $A_4B_4C_4$ is equilateral, implying that $\angle B_4C_4A_4 = \angle C_4A_4B_4 = \angle A_4B_4C_4 = 60^\circ$. Hence $\angle A_0B_4C_4 + \angle A_0C_4B_4 = \angle B_0C_4A_4 + \angle A_0C_4B_4 = 120^\circ$, and so $\angle A_0B_4C_4 = \angle B_0C_4A_4$. Hence $\angle AB_2C_1 = \angle BC_2A_1$, or $\angle B_1B_2C_1 = \angle C_1C_2A_1$. Since the vertex angles of the isosceles triangles $B_1B_2C_1$ and $C_1C_2A_1$ are equal, then two triangles are similar and hence congruent to each other, implying that $C_1B_1 = C_1A_1$. Since $C_1B_1 = C_1A_1$ and $A_2B_1 = A_2A_1$, line C_1A_2 is a perpendicular bisector of triangle $A_1B_1C_1$. Likewise, so are lines A_1B_2 and B_1C_2. Therefore, lines C_1A_2, A_1B_2, and B_1C_2 concur at the circumcenter of triangle $A_1B_1C_1$.

It is not difficult to see that triangle $A_0B_4C_4$ is congruent to triangle $B_0C_4A_4$ (and to triangle $C_0A_4B_4$).

Second Solution:



Let P be a point inside triangle ABC such that triangle A_1A_2P is equilateral. Then $A_1P = A_1A_2 = C_2C_1 = A_1C_2$ and $A_1P \parallel C_2C_1$ ($\angle PA_1A_2 = \angle B = 60^\circ$), and so $A_1PC_1C_2$ is a rhombus. Likewise, $A_2B_1B_2P$ is also a rhombus. Hence triangle B_2C_1P is equilateral. Hence we may set $\angle A_1A_2B_1 = \alpha$,

$\angle A_2B_1B_2 = \angle A_2PB_2 = \beta$, and $\angle C_1C_2A_1 = \angle A_1PC_1 = \gamma$. We have

$$\begin{aligned}\alpha + \beta &= 360^\circ - (\angle B_1A_2C + \angle A_2B_1C) \\ &= 360^\circ - (180^\circ - \angle C) = 240^\circ\end{aligned}$$

and

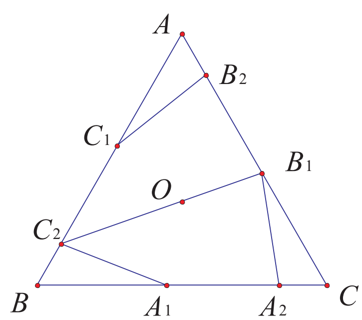
$$\begin{aligned}\gamma + \beta &= \angle A_1PC_1 + \angle A_2PB_2 \\ &= 360^\circ - \angle B_2PC_1 - \angle A_1PA_2 = 240^\circ.\end{aligned}$$

Hence $\angle A_1A_2B_1 = \alpha = \gamma = \angle C_1C_2A_1$. Thus isosceles triangles $A_1A_2B_1$ and $C_1C_2A_1$ are congruent. In exactly the same way, we can show that all three isosceles triangles $A_1A_2B_1$, $B_1B_2C_1$, and $C_1C_2A_1$ are congruent to each other, implying that triangle $A_1B_1C_1$ is equilateral. We can then finish as we did in the first solution.

Third Solution: Consider the six vectors of equal lengths, with zero sum:

$$\mathbf{u} = \overrightarrow{A_2B_1},\ \mathbf{v} = \overrightarrow{B_2C_1},\ \mathbf{w} = \overrightarrow{C_2A_1},$$
$$\mathbf{u}_1 = \overrightarrow{B_1B_2},\ \mathbf{v}_1 = \overrightarrow{C_1C_2},\ \mathbf{w}_1 = \overrightarrow{A_1A_2}.$$

Clearly, vectors $\mathbf{u}_1, \mathbf{v}_1$, and $\mathbf{w}_1$ form a equilateral triangle (by placing tails with heads), and so add up to the zero vector. Consequently, vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ add up to zero vector, or $\mathbf{u} + \mathbf{v} = -\mathbf{w}$.



The sum of two vectors of equal length is a vector of the same length only if they make an 120° angle. (This follows either from the parallelogram interpretation of vector addition or from the **Law of Cosines**.) Therefore the three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ define an equilateral triangle (by placing tails with heads). Hence $\angle AB_2C_1 = \angle BC_2A_1 = \angle CA_2B_1$. Consequently corner triangles AB_2C_1, BC_2A_1, and CA_2B_1 are similar to each other, and in fact congruent to each other, as $B_2C_1 = C_2A_1 = A_2B_1$. Thus the whole configuration is invariant under the rotation centered at O (the circumcenter of triangle ABC) with an angle of 120°.

Because $\angle A_1A_2B_1 = \angle B_1B_2C_2$ and $\angle B_2C_1C_2 = \angle C_1A_1A_2$, line B_1C_2 is a symmetry axis of the hexagon $A_1A_2B_1B_2C_1C_2$, so it must pass through the rotation center O. In conclusion, all three lines in the question concur at O.

Note: From these solutions, we can conclude that equilateral triangles ABC, $A_1B_1C_1$, and $A_2B_2C_2$ share the same center. We can also prove that $A_1A_2B_1B_2C_1C_2$ circumscribes the incircle of triangle ABC, from which the desired follows (by **Brianchon's Theorem**).

2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \ldots, a_n$ leave distinct remainders upon division by n. Prove that every integer occurs exactly once in the sequence.

Solution: The conditions of the problem can be reformulated by saying that for every positive integer n, the numbers $a_1, a_2, \ldots, a_n$ form a complete set of residues modulo n. We proceed our proof as the following.

(1) First, we claim that the sequence consists of distinct integers; that is, if $1 \le i < j$, then $a_i \neq a_j$. Otherwise the set $\{a_1, a_2, \ldots, a_j\}$ would contain at most $j-1$ distinct residues modulo j, violating our new formulation of the conditions of the problem.

(2) Second, we show that numbers in the sequence are fairly close to each other. More precisely, we claim that if $1 \le i < j \le n$, then $|a_i - a_j| \le n-1$. For if $m = |a_i - a_j| \ge n$, then the set $\{a_1, a_2, \ldots, a_m\}$ would contain two numbers congruent modulo m, violating our new formulation of the conditions of the problem.

(3) Third, we show that the set $\{a_1, a_2, \ldots, a_n\}$ contains a block of consecutive numbers. Indeed, for every positive integer n, let i_n and j_n be the indices such that a_{i_n} and a_{j_n} are respectively the smallest and the largest number among $a_1, a_2, \ldots, a_n$. By (2), we conclude that $a_{j_n} - a_{i_n} = |a_{j_n} - a_{i_n}| \le n-1$. By (1), we conclude that $\{a_1, a_2, \ldots, a_n\}$ consists of all integers between a_{i_n} and a_{j_n} (inclusive).

(4) Finally, we show that every integer appears in the sequence. Let x be an arbitrary integer. Because $a_k < 0$ for infinitely many indices k and the terms of the sequence are distinct, it follows that there exists i such that $a_i < x$. Likewise, there exists j such that $x < a_j$. Let n be an integer with $n \ge \max\{i, j\}$. By (3), we conclude that every number between a_i and a_j, including x in particular, is in $\{a_1, a_2, \ldots, a_n\}$. Our proof is thus complete.

3. Let x, y, and z be positive real numbers such that $xyz \ge 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \ge 0.$$

First Solution: Note that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} = 1 - \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2}$$

and its cyclic analogous forms. The given inequality is equivalent to

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \le 3. \qquad (\dagger)$$

In view of the **Cauchy-Schwarz Inequality** and the condition $xyz \ge 1$, we have

$$\begin{aligned}(x^5 + y^2 + z^2)(yz + y^2 + z^2) &\ge \left(x^{\frac{5}{2}}(yz)^{\frac{1}{2}} + y^2 + z^2\right)^2 \\ &\ge (x^2 + y^2 + z^2)^2,\end{aligned}$$

or

$$\frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \le \frac{yz + y^2 + z^2}{x^2 + y^2 + z^2}.$$

Taking the cyclic sum of the above inequality and analogous forms gives

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2}\leq 2+\frac{yz+zx+xy}{x^2+y^2+z^2}.$$

It suffices to show that $xy+yz+zx\leq x^2+y^2+z^2$, which is well known (and can be easily shown by the Cauchy-Schwarz Inequality or $(x-y)^2+(y-z)^2+(z-x)^2\geq 0$).

Second Solution: Given a function f of n variables, we define the symmetric sum

$$\sum_{\text{sym}} f(x_1,\ldots,x_n)=\sum_{\sigma} f(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

where σ runs over all permutations of $1,\ldots,n$ (for a total of $n!$ terms). For example, if $n=3$, and we write x,y,z for x_1,x_2,x_3,

$$\sum_{\text{sym}} x^3 = 2x^3+2y^3+2z^3$$
$$\sum_{\text{sym}} x^2y = x^2y+y^2z+z^2x+x^2z+y^2x+z^2y$$
$$\sum_{\text{sym}} xyz = 6xyz.$$

If $xyz=t^3\geq 1$, set $x=tx_1, y=ty_1$, and $z=tz_1$. Then x_1,y_1, and z_1 are real numbers with $x_1y_1z_1=1$. Note that

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}=\frac{x_1^2+y_1^2+z_1^2}{x_1^5t^3+y_1^2+z_1^2}\leq\frac{x_1^2+y_1^2+z_1^2}{x_1^5+y_1^2+z_1^2}$$

We may further assume $xyz=1$ for inequality (†). We establish inequality (†) in a very mechanical way. Multiplying both sides of the inequality by

$$(x^5+y^2+z^2)(y^5+z^2+x^2)(z^5+x^2+y^2)$$

and canceling the like terms reduces the desired inequality to

$$\sum_{\text{sym}}(x^9+x^3+4x^7y^5)\geq\sum_{\text{sym}}(x^6+x^3y^3+2x^5y^4+2x^4y^2). \qquad (\dagger')$$

By the AM-GM Inequality, we have $x^6+y^6\geq 2x^3y^3$ and $x^9+x^3\geq 2x^6$ and their symmetric analogous forms. Adding them together shows that

$$\sum_{\text{sym}}(x^9+x^3)\geq\sum_{\text{sym}}(x^6+x^3y^3). \qquad (*)$$

It suffice to show that

$$\sum_{\text{sym}} x^7y^5\geq\sum_{\text{sym}} x^5y^4=\sum_{\text{sym}} x^6y^5z \qquad (**)$$

and

$$\sum_{\text{sym}} x^7y^5\geq\sum_{\text{sym}} x^4y^2=\sum_{\text{sym}} x^6y^4z^2. \qquad (***).$$

By the **Weighted AM-GM inequality**, we have

$$5(x^7y^5 + x^5y^7) + (x^7z^5 + x^5z^7) \geq 12x^6y^5z$$

and symmetric analogous forms. Adding them together yields inequality $(**)$.

By the Weighted AM-GM inequality, we have

$$4(x^7y^5 + x^5y^7) + (x^7z^5 + x^5z^7) + (y^7z^5 + y^5z^7) \geq 12x^5y^5z^2$$

and symmetric analogous forms. Adding them together yields inequality $(***)$.

Note: There are ways (other than the combination of inequalities into $(*)$, $(**)$, and $(***)$) of splitting the inequality $(\dagger')$, because the inequality

$$\sum_{\text{sym}} x^7y^5 \geq \sum_{\text{sym}} x^3y^3 = \sum_{\text{sym}} x^5y^5z^2$$

can also be shown by the Weighted AM-GM Inequality.

The weights in establishing $(**)$ and $(***)$ are obtained as follows:

$$\begin{aligned} & 5[7,0,5] + 5[5,7,0] + [7,0,5] + [5,0,7] = 12[6,5,1], \\ & 4[7,5,0] + 4[5,7,0] + 2[7,0,5] + 2[5,0,7] = 12[6,4,2], \\ & 4[7,5,0] + 4[5,7,0] + [7,0,5] + [5,0,7] + [0,5,7] + [0,7,5] \\ = \quad & 4[12,12,0] + [12,0,12] + [0,12,12] = 12[5,5,1]. \end{aligned}$$

Third Solution: (Based on work by Jiayin Kang from China) We shall prove something more, namely that

$$\frac{x^5}{x^5+y^2+z^2} + \frac{y^5}{y^5+z^2+x^2} + \frac{z^5}{z^5+x^2+y^2} \geq 1 \qquad (\ddagger)$$

and

$$1 \geq \frac{x^2}{x^5+y^2+z^2} + \frac{y^2}{y^5+z^2+x^2} + \frac{z^2}{z^5+x^2+y^2}. \qquad (\ddagger')$$

Note that $(\ddagger)$ follows from adding

$$\frac{x^5}{x^5+y^2+z^2} \geq \frac{x^4}{x^4+y^4+z^4} \quad (\text{or } x(y^4+z^4) \geq y^2+z^2)$$

with its analogous cyclic inequalities. By the AM-GM Inequality, we have

$$2x(y^4+z^4) \geq x(y^2+z^2)^2 \geq 2xyz(y^2+z^2) \geq 2(y^2+z^2),$$

as desired.

To establish $(\ddagger')$, we first prove

$$\frac{x^2}{x^5+y^2+z^2} \leq \frac{x}{x+y+z}.$$

As in the second solution, it suffices to prove this inequality for the case in which $xyz = 1$ since replacement of x, y, z by tx, ty, tz, respectively, where $t > 1$ leaves the right-hand side unchanged and decreases the left-hand side. Replacing y^2+z^2 by $xyz(y^2+z^2)$ and simplifying, we see that it suffices to show that

$$x^4 + y^3z + yz^3 \geq x + y + z.$$

By repeated applications of the AM-GM Inequality, we have

$$\begin{aligned}
4(x^4 + y^3z + yz^3) &\geq 4x^4 + 3y^3z + 3yz^3 + 2y^2z^2 \\
&= (2x^4 + y^3z + yz^3) + (x^4 + 2y^3z + y^2z^2) \\
&\quad +(x^4 + 2yz^3 + y^2z^2) \\
&\geq 4x^2yz + 4xy^2z + 4xyz^2 = 4(x + y + z),
\end{aligned}$$

thus confirming that

$$\frac{x^2}{x^5 + y^2 + z^2} \leq \frac{x}{x + y + z}.$$

Adding this and the associated inequalities

$$\frac{y^2}{y^5 + z^2 + x^2} \leq \frac{y}{x + y + z} \quad \text{and} \quad \frac{z^2}{z^5 + x^2 + y^2} \leq \frac{z}{x + y + z},$$

we obtain (‡′).

Fourth Solution: (Based on work by Hyun Soo Kim) We present a third approach of establishing inequality (†). Because $y^2 + z^2 \geq 2yz$ and $xyz \geq 1$, we have

$$\frac{1}{x^5 + y^2 + z^2} \leq \frac{1}{\frac{x^4}{yz} + y^2 + z^2} \leq \frac{1}{\frac{2x^4}{y^2+z^2} + y^2 + z^2}$$

and the cyclic analogous forms. Thus it suffices to show that

$$\frac{x^2 + y^2 + z^2}{\frac{2x^4}{y^2+z^2} + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{\frac{2y^4}{z^2+x^2} + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{\frac{2z^4}{x^2+y^2} + x^2 + y^2} \leq 3.$$

However, since this is a homogeneous inequality, the condition $xyz \geq 1$ is not relevant anymore. Furthermore, we may assume that $x^2 + y^2 + z^2 = 3$. Then the inequality reduces to

$$\frac{1}{\frac{2x^4}{3-x^2} + 3 - x^2} + \frac{1}{\frac{2y^4}{3-y^2} + 3 - y^2} + \frac{1}{\frac{2z^4}{3-z^2} + 3 - z^2} \leq 1,$$

or

$$\frac{3 - x^2}{3x^4 - 6x^2 + 9} + \frac{3 - y^2}{3y^4 - 6y^2 + 9} + \frac{3 - z^2}{3z^4 - 6z^2 + 9} \leq 1,$$

where x, y, and z are positive real numbers with $x^2 + y^2 + z^2 = 3$.

Because $3x^4 - 6x^2 + 9 = 3(x^2 - 1)^2 + 6 \geq 6$ and $3 - x^2 = y^2 + z^2 \geq 0$, we obtain

$$\frac{3 - x^2}{3x^4 - 6x^2 + 9} \leq \frac{3 - x^2}{6}.$$

Adding the above inequality and the cyclic analogous forms gives

$$\begin{aligned}
&\frac{3 - x^2}{3x^4 - 6x^2 + 9} + \frac{3 - y^2}{3y^4 - 6y^2 + 9} + \frac{3 - z^2}{3z^4 - 6z^2 + 9} \\
\leq\ & \frac{9 - (x^2 + y^2 + z^2)}{6} = 1,
\end{aligned}$$

as desired.

Fifth Solution: (Based on work by Xuancheng Shao from China) We claim that

$$\frac{1}{x^5+y^2+z^2} \le \frac{\frac{3}{2}\cdot(y^2+z^2)}{(x^2+y^2+z^2)^2}.$$

Adding the above inequality and the cyclic analogous forms yields the desired inequality (†). Because $xyz \ge 1$, $x \ge \frac{1}{yz}$, and so

$$\frac{1}{x^5+y^2+z^2} \le \frac{1}{\frac{x^4}{yz}+y^2+z^2},$$

or

$$\frac{1}{x^5+y^2+z^2} \le \frac{yz}{x^4+yz(y^2+z^2)}.$$

It suffices to show that

$$\frac{2yz}{x^4+yz(y^2+z^2)} \le \frac{3(y^2+z^2)}{(x^2+y^2+z^2)^2},$$

or

$$2yz(x^2+y^2+z^2)^2 \le 3x^4(y^2+z^2)+3yz(y^2+z^2)^2.$$

Expanding the left-hand side of the last inequality in x^2 and y^2+z^2 gives

$$2x^4yz+4x^2yz(y^2+z^2) \le 3x^4(y^2+z^2)+yz(y^2+z^2)^2.$$

Because $3x^4(y^2+z^2) \ge 6x^4yz$, it suffices to show that

$$4x^2yz(y^2+z^2) \le 4x^4yz+yz(y^2+z^2)^2,$$

which is evident by the AM-GM Inequality.

Sixth Solution: (Based on work by Iurie Boreico from Moldova) Note that

$$\frac{x^5-x^2}{x^5+y^2+x^2} \ge \frac{x^5-x^2}{x^3(x^2+y^2+z^2)}$$

is equivalent to

$$\frac{(x^3-1)^2(y^2+z^2)}{x(x^5+y^2+z^2)(x^2+y^2+z^2)} \ge 0,$$

which is true for all positive x, y, z. Hence

$$\frac{x^5-x^2}{x^5+y^2+z^2} \ge \frac{x^2-\frac{1}{x}}{x^2+y^2+z^2}.$$

Summing the above inequality with its analogous cyclic inequalities, we see that the desired result follows from

$$x^2+y^2+z^2-\frac{1}{x}-\frac{1}{y}-\frac{1}{z} \ge 0.$$

Since $xyz \geq 1$,

$$\begin{aligned} & x^2+y^2+z^2-\frac{1}{x}-\frac{1}{y}-\frac{1}{z} \\ = \ & x^2+y^2+z^2-\frac{yz+xz+xy}{xyz} \\ \geq \ & x^2+x^2+z^2-yz-xz-xy \\ = \ & \frac{(x-y)^2+(y-z)^2+(z-x)^2}{2} \geq 0, \end{aligned}$$

so we are done.

4. Consider the sequence $a_1, a_2, \ldots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1.$$

for all positive integers n. Determine all positive integers that are relatively prime to every term of the sequence.

Solution: The answer is that 1 is the only such number. It suffices to show that every prime p divides a_n for some positive integer n. Note that both $p = 2$ and $p = 3$ divide $a_2 = 2^2+3^2+6^2-1 = 48$. Now we assume that $p \geq 5$. By **Fermat's Little Theorem**, we have $2^{p-1} \equiv 3^{p-1} \equiv 6^{p-1} \equiv 1 \pmod{p}$. Then

$$3 \cdot 2^{p-1} + 2 \cdot 3^{p-1} + 6^{p-1} \equiv 3 + 2 + 1 \equiv 6 \pmod{p},$$

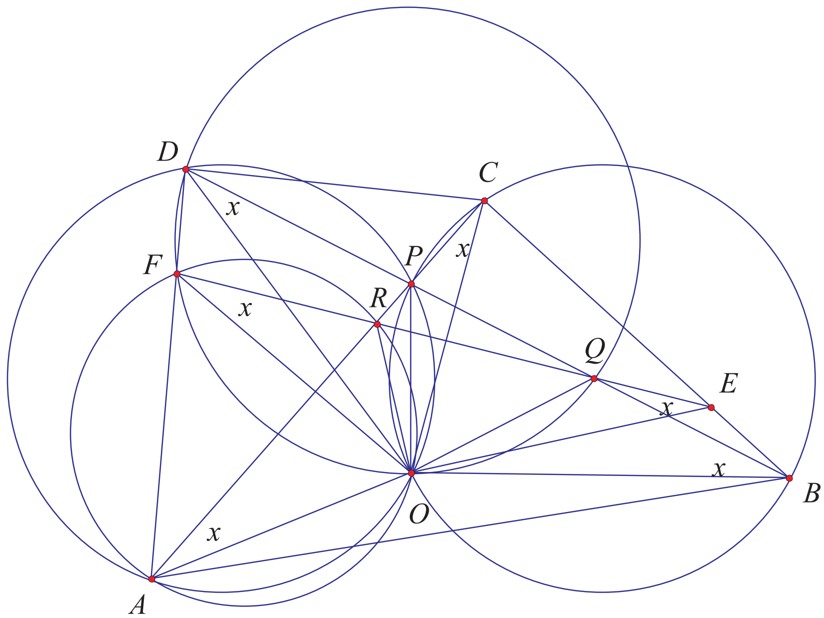
or, $6(2^{p-2} + 3^{p-2} + 6^{p-2} - 1) \equiv 0 \pmod{p}$; that is, $6a_{p-2}$ is divisible by p. Because p is relatively prime to 6, a_{p-2} is divisible by p, as desired.

5. Let $ABCD$ be a given convex quadrilateral with sides BC and AD equal in length and not parallel. Let points E and F lie on sides BC and AD, respectively, such that $BE = DF$. Lines AC and BD meet at P, lines BD and EF meet at Q, and lines EF and AC meet at R. Consider all the triangles PQR as E and F vary. Show that the circumcircles of these triangles have a common point other than P.

First Solution: We consider the configuration shown below. Note that this argument has more than one configuration, since O can be above P. As written, the argument works under the assumption that all angles are taken to be directed modulo π (or 180°). If the reader is not familiar, please try to develop a similar proof by relabeling $ABCD$ as $CDAB$.

Let ω_1 and ω_2 denote the circumcircles of triangles ADP and BCP, respectively. Because $\frac{AD}{\sin\angle DPA} = \frac{BC}{\sin\angle BPC}$, by the **Extended Law of Sines**, ω_1 and ω_2 have the same size. Let R be the radius of ω_1 and ω_2. Let O be the second intersection (other than P) of ω_1 and ω_2. (Because $AD = BC$ and $AD \nparallel BC$, this point is well defined.) Again, applying the Extended Law of Sines in triangles CPO and APO gives $2R = \frac{PO}{\sin\angle PCO} = \frac{PO}{\sin\angle OAP}$, and so $\sin\angle ACO = \sin\angle OAC$.

Since $\angle ACO$ and $\angle OAC$ are equal angles in triangle ACO, the triangle is isosceles triangle with $AO = CO$. Likewise, $\angle DBO = \angle ODB$ and BDO is an isosceles triangle with $BO = DO$. Because $ADPO$ is cyclic, $\angle BDO = \angle PDO = \angle PAO = \angle CAO$. Thus triangle ACO is similar to triangle DBO. Consequently, we have $\angle COA = \angle BOD$. Consider a rotation **H** centered at O that sends A



to C; that is, $\mathbf{H}(A) = C$. Then $\mathbf{H}(D) = B$. Thus $\mathbf{H}$ sends triangle ADO to triangle CBO. Because $DF = BE$, it follows that $\mathbf{H}(F) = E$, and so $\angle EOF = \angle COA = \angle BOD$ and $EO = FO$. It follows that

$$\angle OFR = \angle OFQ = \angle ODQ = \angle OAC = \angle OAR = x, \tag{$*$}$$

implying that quadrilaterals $DFOQ$ and $AFRO$ are cyclic. Because $AFRO$ and $AOPD$ are cyclic, we have

$$\angle ROF = \angle RAF = \angle PAD = \angle POD,$$

or

$$\angle DOF = \angle POR.$$

Because $DFOQ$ is cyclic, $\angle DQF = \angle DOF$. Combining the last two equations gives $\angle PQR = \angle DQF = \angle DOF = \angle POR$; that is $PQOR$ is cyclic.

Note: There are many cyclic quadrilaterals in the figure. For example, we can also finish the proof by noting

$$\angle OQB = \angle OEB = \angle ORC = \angle ORP$$

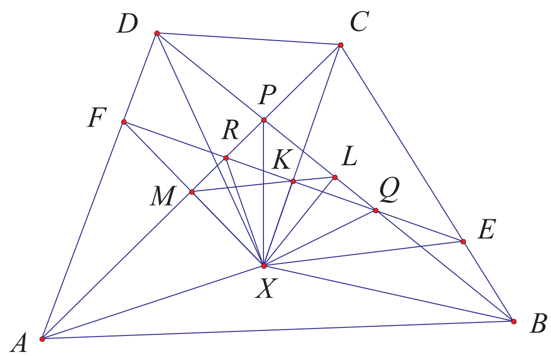
because $BEQO$ and $OECR$ are cyclic.

Second Solution: Let the perpendicular bisectors of segments AC and BD meet at X. We show that the circumcircles of triangles PQR pass through X, which is fixed. (Because $AD = BC$ and $AD \not\parallel BC$, this point is well defined.)

Because $XA = XC$, $XB = XD$, and $DA = BC$, it follows that isosceles triangles XDA and XBC are congruent, with F and E being corresponding points. Let $\mathbf{H}_1$ denote the rotation centered at X that sends A to C. Then $\mathbf{H}_1(D) = B$ and $\mathbf{H}_1(F) = E$. This implies that $XE = XF$ and

$$\angle EXF = \angle BXD = \angle CXA,$$

which is equal to the angle of rotation. Therefore, isosceles triangles EXF, BXD, and CXA are similar to each other.



Denote by K, L, and M the feet of perpendiculars from X to lines EF, BD, and CA, respectively. In view of the similarity just mentioned, we have

$$\frac{XK}{XE} = \frac{XL}{XB} = \frac{XM}{XC} = \lambda$$

and $\angle EXK = \angle BXL = \angle CXM = \alpha$. Let $\mathbf{S}$ denote the rotation centered at X through angle α, composed with the homothety centered at X with ratio λ. (Hence $\mathbf{S}$ is a **spiral similarity**.) Then $\mathbf{S}$ takes points B, E, and C to points L, K, and M, respectively, implying that points L, K, and M are collinear.

Because $\angle XMR = \angle XKR = \angle XMP = \angle XLP = 90^\circ$, quadrilaterals $XKRM$ and $XLPM$ are cyclic, implying that

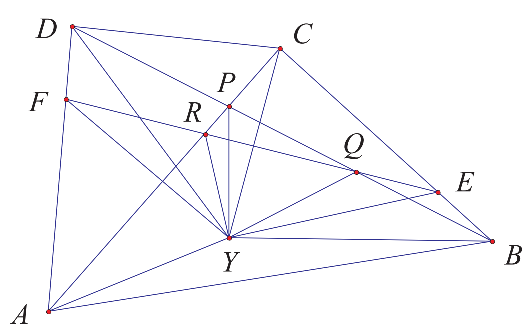
$$\angle XRQ = \angle XRK = \angle XMK = \angle XML = \angle XPL = \angle XPQ.$$

Hence $XQPR$ is cyclic.

Third Solution: (Composition of the work by Sherry Gong and Thomas Mildorf) Applying the Law of Sines to triangles ARF and CRE gives

$$\frac{AR}{RC} = \frac{AR}{AF} \cdot \frac{CE}{CR} = \frac{\sin \angle AFR}{\sin \angle ARF} \cdot \frac{\sin \angle CRE}{\sin \angle CER} = \frac{\sin \angle AFR}{\sin \angle CER},$$

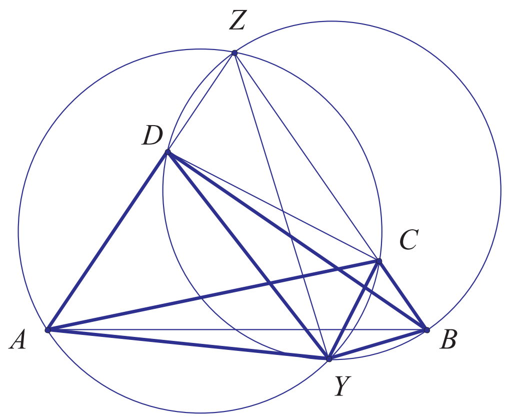
as $\angle ARF = \angle CRE$.



Likewise,

$$\frac{DQ}{QB} = \frac{\sin \angle DFQ}{\sin \angle BEQ} = \frac{\sin \angle AFR}{\sin \angle CER} = \frac{AR}{RC},$$

by noting that $\angle DFQ + \angle AFR = 180^\circ$ and $\angle BEQ + \angle CER = 180^\circ$. Let Y be the center of the spiral similarity (denoted by $\mathbf{S}_1$) that sends segment BD to CA. (The existence of this center is to be explained later). Then $\mathbf{S}_1(Q) = R$. Then we have $\angle BPC = \angle QYR$, because both are the angle of rotation of $\mathbf{S}_1$. Hence $RPQY$ is cyclic; that is, the circumcircle of triangle PQR always passes through Y.



Now we consider the existence of point Y. For any two nonparallel segments AD and BC (not necessarily having equal length), let Z be the intersection of lines AD and BC. Then Y is the second intersection of circumcircles of triangles ACZ and BDZ. (Because these two circles clearly are not tangent at Z, point Y exists.) Indeed, from the cyclic quadrilaterals $BYDZ$ and $AZCY$, we have $\angle CBY = \angle ZBY = \angle ADY$ and $\angle YCB = \angle YAZ = \angle YAD$, implying that triangle ADY is similar to CBY; that is, Y is the center of spiral similarity that sends triangle ADO to triangle CBO.

Note: Combining the three proofs, we note that $O = X = Y$ and $\mathbf{S} = \mathbf{S_1}$. Certain parts of the three proofs are interchangeable. Also, in the second solution, the line passing through points K, L, and M is the **Simson line** of X with respect to triangle PQR.

6. In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than $\frac{2}{5}$ of the contestants. Nobody solved all 6 problems. Show that there are at least 2 contestants who each solved exactly 5 problems each.

First Solution: Suppose that there were n contestants. Let p_{ij}, with $1 \le i < j \le 6$, be the number of contestants who solved problems i and j, and let n_r, with $0 \le r \le 6$, be the number of contestants who solved exactly r problems. Clearly, $n_6 = 0$ and $n_0 + n_1 + \cdots + n_5 = n$.

By the given condition, $p_{ij} > \frac{2n}{5}$, or $5p_{ij} > 2n$. Hence $5p_{ij} \ge 2n+1$, or $p_{ij} \ge \frac{2n+1}{5}$. We define the set

$$U = \{(c, \{i, j\}) \mid \text{ contestant } c \text{ solved problems } i \text{ and } j\}.$$

If we compute $|U|$, the number of elements in U, by summing over all pairs $\{i, j\}$, we have

$$|U| = \sum_{1 \le i < j \le 6} p_{ij} \ge 15 \cdot \frac{2n+1}{5} = 6n + 3 = 6(n_0 + n_1 + \cdots n_5) + 3.$$

A contest who solved exactly r problems contributes a "1" to $\binom{r}{2}$ summands in this sum (where $\binom{r}{2} = 0$ for $r < 2$), if we compute $|U|$ by summing over all contestants c. Therefore,

$$|U| = \sum_{r=0}^{6} \binom{r}{2} n_r = n_2 + 3n_3 + 6n_4 + 10n_5.$$

It follows that $n_2 + 3n_3 + 6n_4 + 10n_5 \geq 6(n_0 + n_1 + \cdots + n_5) + 3$, or

$$4n_5 \geq 3 + 6n_0 + 6n_1 + 5n_2 + 3n_3 \geq 3,$$

implying that $n_5 \geq 1$. We need to show that $n_5 \geq 2$. We approach indirectly by assuming that $n_5 = 1$. We call this person the *winner* (denoted by W), and without loss of generality, we may assume that the winner failed to solve problem 6. Then $n_0 = n_1 = n_2 = n_3 = 0$. Hence $n_4 = n - 1$, and so

$$|U| = n_2 + 3n_3 + 6n_4 + 10n_5 = 6n + 4 > 6n + 3 = 15 \cdot \frac{2n+1}{5}.$$

It follows that $p_{ij} = \frac{2n+1}{5}$ for 14 out of the 15 total pairs (i, j) with $1 \leq i < j \leq 15$, and for the remaining pair (s, t), $p_{st} = \frac{2n+1}{5} + 1 = \frac{2n+6}{5}$. Without loss of generality we may assume that $1 < s < t \leq 6$. (This is because that the only assumption we had was that the winner failed to solve problem 6. Hence problems 1 through 5 are equally important.)

First we consider the sum

$$u_1 = p_{12} + p_{13} + p_{14} + p_{15} + p_{16} = 5 \cdot \frac{2n+1}{5} = 2n + 1,$$

because $p_{1k} \neq p_{st}$. Suppose that problem 1 was solved by x contestants $c_1, c_2 \ldots, c_x$ other than the winner. Each of these contestants c_i solved 3 problems other than problem 1. It follows that each of these contestants contributed a "3" to the sum u_1. The winner contributed a "4" to the sum u_1. Hence $u_1 = 3x + 4$. The pair p_{st} does not appear as a summand for u_1. Thus $u_1 = 3x + 4 = 2n + 1$, implying that n is divisible by 3.

Second we consider the sum

$$\begin{aligned} v_6 &= p_{16} + p_{26} + p_{36} + p_{46} + p_{56} \\ &= \begin{cases} 2n+1 & \text{if } p_{k6} \neq p_{st} \text{ for all } 1 \leq k \leq 5, \\ 2n+2 & \text{if } p_{k6} = p_{st} \text{ for some } 1 \leq k \leq 5. \end{cases} \end{aligned}$$

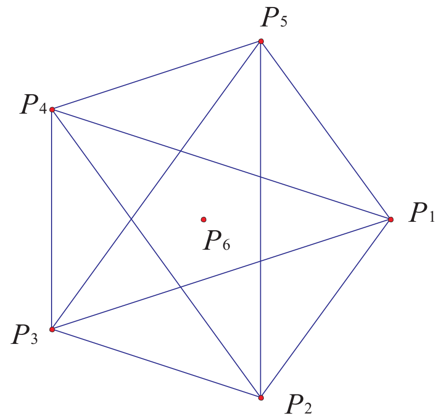
Suppose that problem 6 was solved by y contestants $d_1, d_2 \ldots, d_y$. The winner was not among them, and each of these contestants solved 3 problems other than problem 6. It follows that each of these contestants contributed a "3" to the sum v_6, and so

$$v_6 = 3y = \begin{cases} 2n+1, \\ 2n+2. \end{cases}$$

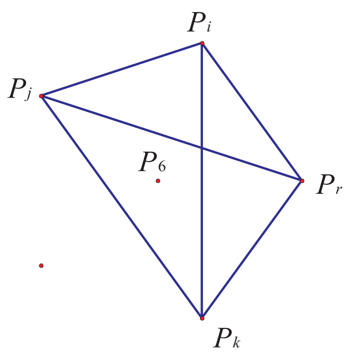
In either case, we conclude that n is not divisible by 3, which is a contradiction of our previous observation. Therefore, our assumption $n_5 = 1$ was wrong, and so $n_5 \geq 2$, as desired.

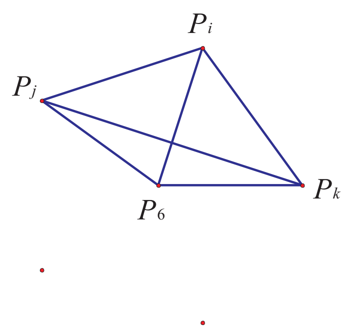
Second Solution: We maintain the same notation as the first solution. We define a graph with each of the six problems as a vertex, and we construct an edge between a pair of problems if this problem is solved by a contestant. This is a multi-graph; that is, multiple edges are allowed between a pair of vertices. The number of edges between a pair of vertices P_i and P_j is equal to the number of contestants who solved this pair of problems.

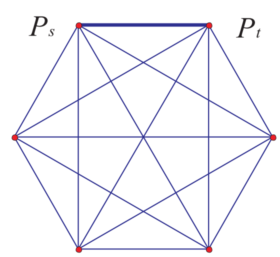
We construct our graph one contestant at time. First, we build it with our winner. Clearly, we have the following graph.



Second, we add contestants one at a time. Each of them solved exactly 4 problems. Hence each of them adds a K_4 (complete graph of 4 vertices) as shown below. (Two possibilities: if the contestant solved problem 6 or not.) In any case, the degree of each vertex either increases by 3 or remains the same. We conclude that, after we put the winner into the graph, the degree of each vertex is invariant modulo 3. In particular, after all of the students are added to the graph, five vertices have the same degree modulo 3 and the sixth vertex has a different degree modulo 3.







On the other hand, as we have shown in the first solution, $p_{ij} = \frac{2n+1}{5}$ for 14 out of the 15 total pairs (i, j) with $1 \leq i < j \leq 15$, and for the remaining pair (s, t), $p_{st} = \frac{2n+1}{5} + 1 = \frac{2n+6}{5}$. Thus, modulo 3, our graph should be in the following form: 14 of the edges occur with multiplicity of the same residue modulo 3 and the other $(P_s P_t)$ occurs with a multiplicity which is a different residue. However, this would mean that two vertices have the same degree modulo 3 and the other four vertices have a different degree modulo 3, which is a contradiction.

Third Solution: In this solution, we incorporate the idea of the second solution in combinatorial computations. Let $m = \frac{2n+1}{5}$. As shown in the first solution, $p_{ij} = m$ for all but one pair, namely $\{s, t\}$ where $p_{st} = m + 1$.

Let

$$d_i = \sum_{j \neq i} p_{ij}, \quad i = 1, 2, \ldots, 6.$$

We have just seen that $d_s = d_t = 5m + 1$ and $d_i = 5m$ otherwise. On the other hand, consider what happens if we build up the 6-tuple $(d_1, d_2, \ldots, d_6)$ one contestant at a time, starting with W. Thus we start with $(4, 4, 4, 4, 4, 0)$, and every subsequent contestant adds a permutation of $(3, 3, 3, 3, 0, 0)$. Thus

$$(d_1, d_2, \ldots, d_6) \equiv (1, 1, 1, 1, 1, 0) \pmod 3,$$

contradicting the earlier conclusion that $d_s = d_t = 5m + 1$ and $d_i = 5m$ otherwise. Hence there were are least two persons to solve five problems.

Fourth Solution: (Based on the work of Sherry Gong) We suppose, for the sake of contradiction, that there exists a counterexample to the problem statement, and we consider a counterexample scenario with a minimal number n of students. We can add solutions to students' results until one student, the winner, solved 5 problems, and the rest of the students solved 4 problems; since we have only increased the number of students solving any pair, this is still a counterexample. If $n \leq 2$, there is a problem not solved by the first student and a problem not solved by the second student, and thus a pair solved by no one; thus $n > 2$.

If every 4-tuple of problems was solved by a non-winning contestant, then we can remove $\binom{6}{4} = 15$ contestants, one who solved each 4-tuple. Each pair of problems will now have been solved by 6 ($\binom{4}{2}$, the number of 4-tuples a given pair of problems is in) fewer contestants, but there are 15 fewer contestants overall. Since $\frac{2}{5}15 = 6$, we will still have a situation where each pair is solved by more than $\frac{2}{5}$ of the contestants, but only one student solved 5 problems, which contradicts our choice of a counterexample with a minimal number of students.

We now consider a multi-graph, as in the second solution, where the vertices are problems, and each edge corresponds to a student having solved the problems at its ends. So suppose no non-winning contestant solved the four problems A, B, C, D and let the other two problems be E and F. Let the edges among $\{A, B, C, D\}$ be in a set S, and S' be the set S plus the additional edge EF. Let the edges not in S' comprise the set T. Any non-winner who solves E and F contributes 2 edges to S' and 4 edges to T, and any other non-winner contributes 3 edges each to S' and T.

As we have shown in the first solution, $p_{ij} = \frac{2n+1}{5}$ for 14 out of the 15 total pairs (i, j) with $1 \leq i < j \leq 15$ and $p_{i_0 j_0}$, for exactly one pair (i_0, j_0), is equal to $\frac{2n+1}{5} + 1 = \frac{2n+6}{5}$. So since T contains edges between 8 pairs and S' contains edges between 7 pairs, $|T| - |S| \leq \frac{2n+1}{5} + 1$.

If the winner solves E and F, she contributes 4 edges to S' and 6 edges to T. So we see that students solving E and F contribute 2 more edges to T than S' and other students contribute the same amount to T and S'. Since at least $\frac{2n+1}{5}$ students solve E and F, we know $|T| - |S'| \geq 2 \cdot \frac{2n+1}{5}$. Thus $\frac{2n+1}{5} + 1 \geq 2 \cdot \frac{2n+1}{5}$, which implies $n \leq 2$, a contradiction.

We conclude that the winner does not solve both E and F, and she contributes 6 edges to S' and 4 edges to T. Since at least $\frac{2n+1}{5}$ students solve E and F, we know $|T| - |S'| \geq 2 \cdot \frac{2n+1}{5} - 2$. Thus $\frac{2n+1}{5} + 1 \geq 2 \cdot \frac{2n+1}{5} - 2$, which implies $n \leq 7$. We also see that if no non-winning contestant solves a certain 4-tuple, then the winner must solve the problems in the 4-tuple. However, there are at most 6 non-winners, who solve at most 6 of the 4-tuples, and the winner solves 5 of the 4-tuples, which is a contradiction because there are 15 total 4-tuples.

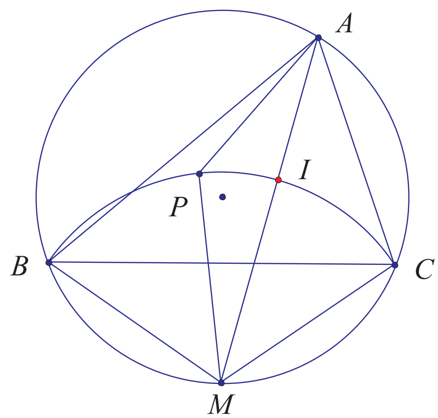
6 IMO 2006

1. Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Solution: We begin by proving a well-known fact.



Lemma. *Let ABC be a triangle with circumcenter O, circumcircle γ, and incenter I. Let M be the second intersection of line AI with γ. Then M is the circumcenter of triangle IBC.*

Proof: Let $\angle A = 2\alpha, \angle B = 2\beta$. Note that M is on the opposite side of line BC as A. We have $\angle CBM = \angle CAM = \alpha$, so that $\angle IBM = \angle IBC + \angle CBM = \beta + \alpha$. Also, $\angle BIM = \angle BAI + \angle ABI = \alpha + \beta$. Thus, triangle IBM is isosceles with $BM = IM$. Similarly, $CM = IM$. This proves the claim. ∎

Back to our current problem, we note that

$$(\angle PBA + \angle PCA) + (\angle PBC + \angle PCB) = \angle B + \angle C,$$

so

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB = \frac{1}{2}(\angle B + \angle C).$$

In triangles PBC, we have

$$\angle BPC = 180^\circ - (\angle PBC + \angle PCB) = 180^\circ - \frac{1}{2}(\angle B + \angle C).$$

It is clear that $\angle IBC + \angle ICB = \frac{1}{2}(\angle B + \angle C)$, and so in triangle BCI,

$$\angle BIC = 180^\circ - \frac{1}{2}(\angle B + \angle C).$$

We conclude that $\angle BPC = \angle BIC$; that is, points B, C, I, and P lie on a circle. By the Lemma, they all lie on a circle centered at M. In particular, we have $MP = MI$.

In triangle APM, we have

$$AI + IM = AM \leq AP + PM = AP + IM,$$

implying that $AI \leq AP$. Equality holds if and only if $AM = AP + PM$; that is, A, P, and M are collinear, or $P = I$.

2. Let $\mathcal{P}$ be a regular 2006-gon. A diagonal of $\mathcal{P}$ is called *good segment* if its endpoints divide the boundary of $\mathcal{P}$ into two parts, each composed of an odd number of sides of $\mathcal{P}$. The sides of $\mathcal{P}$ are also called *good segment*.

   Suppose $\mathcal{P}$ has been dissected into triangles by 2003 diagonals, no two of which have a common point in the interior of $\mathcal{P}$. Find the maximum number of isosceles triangles having two *good segments* that could appear in such a configuration.

Note: Let M denote the maximum we are looking for. The answer is $M = 1003$.

Let $P = P_1P_2 \ldots P_{2006}$, and let ω denote the circumcircle of $\mathcal{P}$. Without loss of generality, points $P_1, \ldots, P_{2006}$ are arranged in clockwise direction along ω. Then P_iP_j is good if and only if $i - j$ is odd. We call an isosceles triangle (in $\mathcal{T}$) *good* if it has two good segments. Since 2006 is even, a good triangle have exactly two good sides. Any set of 2003 diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a *triangulation* of $\mathcal{P}$ into 2004 triangles. Let $\mathcal{T}$ denote such a triangulation.

It is easy to see that $M \geq 1003$. We can first use diagonals $P_1P_3, P_3P_5, \ldots, P_{2003}P_{2005}$, and $P_{2005}P_1$ to obtain 1003 good triangles. We can then complete the triangulation easily by a triangulations of $P_1P_3 \cdots P_{2005}$ using 1001 diagonals. We present two solutions showing that $M \leq 1003$.

Let $\widehat{P_iP_j}$ denote the directed (clockwise direction) broken line segment $P_iP_{i+1} \ldots P_j$ (where $P_{2006+k} = P_k$). We say $\widehat{P_iP_j}$ is non-major if it contains at most 1003 sides of $\mathcal{P}$.

First Solution: We start with the following lemma.

Lemma *Let P_iP_j is a diagonal used in $\mathcal{T}$, and $\widehat{P_iP_j}$ is non-major and contains n segments of $\mathcal{P}$, then there are at most $\left\lfloor \frac{n}{2} \right\rfloor$ good triangles with vertices on $\widehat{P_iP_j}$. More precise there are at most*

$$\begin{cases} \left\lfloor \frac{j-i}{2} \right\rfloor, & \text{if } i < j \\ \left\lfloor \frac{j-i+2006}{2} \right\rfloor, & \text{if } i > j \end{cases}$$

good triangles with vertices on $\widehat{P_iP_j}$

Proof: Without loss of generality, we may assume that $i < j$. We induct on n.

The bases cases for $n = 1$ and $n = 2$ are trivial. Assume the statement is true for n with $n \leq k$ and $2 \leq k < 1003$. We consider the case $n = k + 1$.

Let $P_iP_aP_j$ be a triangle in $\mathcal{T}$ with P on $\widehat{P_iP_j}$. (Note that P_i, P_a, and P_j lie on non-major arc $\widehat{P_iP_j}$ on ω in clockwise order. By the induction hypothesis, there are at most

$$\left\lfloor \frac{a-i}{2} \right\rfloor \leq \frac{a-i}{2}$$

good triangles with vertices on $\widehat{P_iP_a}$. Similar result holds for $\widehat{P_aP_j}$.

Because $P_iP_aP_j$ is a triangle in $\mathcal{T}$, we conclude that if a good triangles has its vertices on $\widehat{P_iP_j}$ then either it is $P_iP_aP_j$, or all its vertices are on exactly one of $\widehat{P_iP_a}$ or $\widehat{P_aP_j}$. We can now apply the induction hypothesis $\widehat{P_iP_a}$ and $\widehat{P_aP_j}$. We conclude that there are at most

$$1 + \frac{a-i}{2} + \frac{j-a}{2} = \frac{j-i}{2} + 1 \tag{‡}$$

good triangles with vertices on $\widehat{P_iP_j}$.

To finish our proof, we need to reduce the value of the right-hand side of (‡) by 1. We consider the following two cases.

In the first case, we assume that $P_iP_aP_j$ is not good. The summand 1 on the right-hand of (†) should be taken out, and we are done.

In the second case, we assume that $P_iP_aP_j$ is good. Since $\widehat{P_iP_j}$ is non-major, $P_iP_j > P_iP_a$ and $P_iP_j > P_aP_j$. We must have P_iP_a and P_aP_j must be the two equal good sides, and both must be the good sides. Hence both $a - i$ and $j - a$ are odd, and so we can improve (†) to

$$\left\lfloor \frac{a-i}{2} \right\rfloor \le \frac{a-i}{2} - \frac{1}{2},$$

and similar result hold for $\widehat{P_aP_j}$. Then (‡) can be improved to

$$1 + \frac{a-i}{2} - \frac{1}{2} + \frac{j-a}{2} - \frac{1}{2} = \frac{j-i}{2},$$

completing our induction. ■

Since $\widehat{P_iP_j}$ is non-major, $P_aP_b < P_aP_c$ and $P_bP_c < P_aP_c$. Since $P_aP_bP_c$ is good, we must have P_aP_b and P_bP_c be the good segments (with equal lengths). Thus $b - a$ and $c - b$ are both odd. By the induction hypothesis, there are at most

$$\left\lfloor \frac{b-a}{2} \right\rfloor = \frac{b-a}{2} - \frac{1}{2}$$

good triangles with vertices on $\widehat{P_aP_b}$. Similar result holds for $\widehat{P_bP_c}$.

Now we prove our main result. Let P_iP_k be the longest diagonal used in $\mathcal{T}$. Let $P_iP_jP_k$ be a non-obtuse triangle in $\mathcal{T}$. Without loss of generality, we may assume that $i < j < k$. Since $P_iP_jP_k$ is non-obtuse, $\widehat{P_iP_j}$, $\widehat{P_jP_k}$, and $\widehat{P_kP_i}$ are all non-major. By the lemma, there are at most

$$\begin{aligned} & \left\lfloor \frac{j-i}{2} \right\rfloor + \left\lfloor \frac{k-j}{2} \right\rfloor + \left\lfloor \frac{i-k+2006}{2} \right\rfloor \\ \le \;& \frac{j-i}{2} + \frac{k-j}{2} + \frac{i-k+2006}{2} = 1003 \end{aligned}$$

good triangles besides $P_iP_jP_k$.

If $P_iP_jP_k$ is not good, we are done. If it is, then exactly two of $j-i$, $k-j$, and $i-k$ are odd, and so $(*)$ is strict inequality. We still have at most $1002 + 1 = 1003$ good triangles in this case, completing our proof.

Second Solution: Let $P_iP_jP_k$ $(i < j < k)$ be a good triangle, with P_iP_j and P_jP_k being good segments. This means that there are an odd number of sides of $\mathcal{P}$ between P_i and P_j and also between P_j and P_k. We say $\widehat{P_iP_j}$ and $\widehat{P_jP_k}$ *belong to* triangle ABC.

At least one side in each of these groups does not belong to any other good triangle. This is so because any odd triangle whose vertices are among the points between P_i and P_j has two sides of equal length and therefore has an even number of sides belonging to it in total. Eliminating all sides belonging to any other good triangle in $\widehat{P_iP_j}$ must therefore leave at least one side that belongs to no other good triangle. Same argument applies to $\widehat{P_jP_k}$. Let us assign these two sides (one in $\widehat{P_iP_j}$ and one in $\widehat{P_jP_k}$) to triangle $P_iP_jP_k$.

To each good triangle we have thus assigned a pair of sides, with no two good triangles sharing an assigned side. It follows that at most 1003 good triangles can appear in the triangulation; that is, $M \le 1003$.

3. Determine the least real number M such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \le M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b, and c.

Note: Consider polynomial

$$P(a,b,c) = ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2).$$

It is not difficult to check that $P(a,a,c)=0$. Hence $a-b$ divides $P(a,b,c)$. Since $P(a,b,c)$ is cyclic symmetric, we conclude that $(a-b)(b-c)(c-a)$ divides $P(a,b,c)$. Since $P(a,b,c)$ is a cyclic homogenous polynomial of degree 4 (each monomial in expansion of $P(a,b,c)$ has degree 4) and $(a-b)(b-c)(c-a)$ is cyclic homogenous polynomial of degree 3,

$$P(a,b,c) = (a-b)(b-c)(c-a)Q(x),$$

where $Q(x)$ is a cyclic homogenous polynomial of degree 1; that is, $Q(x) = k(a+b+c)$ for some constant k. It is easy to deduce that $k=1$ and

$$P(a,b,c) = (a-b)(b-c)(c-a)(a+b+c).$$

The given inequality now reads

$$|(a-b)(b-c)(c-a)(a+b+c)| \le M(a^2+b^2+c^2)^2. \qquad (*)$$

Since the above inequality is symmetric with respect to a, b, and c, we may assume that $a \ge b\ gec$. (Indeed, we may assume that $a > b > c$, because otherwise the left-hand side of $(*)$ is 0, and we have nothing to prove.) Thus $(*)$ reduce to

$$(a-b)(b-c)(a-c)(a+b+c) \le M(a^2+b^2+c^2)^2 \qquad (**)$$

for real numbers $a > b > c$. Note also that $(**)$ is homogenous (of degree 4). We may further assume that $a+b+c=1$. Then $(**)$ reduce to

$$(a-b)(b-c)(a-c) \le M(a^2+b^2+c^2)^2 \qquad (\dagger)$$

for real numbers $a > b > c$ with $a+b+c=1$. Setting $a-b=x$ and $b-c=y$, we have $a-c=x+y$. Note that

$$\begin{aligned}
& (a-b)^2+(b-c)^2+(c-a)^2 \\
= \;& 2(a^2+b^2+c^2)-2(ab+bc+ca) \\
= \;& 2(a^2+b^2+c^2)-[(a+b+c)^2-(a^2+b^2+c^2)] \\
= \;& 3(a^2+b^2+c^2)-1.
\end{aligned}$$

We can rewrite (†) as

$$9xy(x+y) \le M[x^2+y^2+(x+y)^2+1]^2 \tag{‡}$$

for positive real numbers x and y. It suffice to find the least M satisfying (‡). There are many ways to finish. We present two typical ones.

First Solution: We rewrite (‡) as

$$\begin{aligned} \frac{9(x+y)}{M} &\le \left(\frac{x^2+y^2+(x+y)^2+1}{\sqrt{xy}}\right)^2 \\ &= \left(\frac{2(x+y)^2+1}{\sqrt{xy}} - 2\sqrt{xy}\right)^2. \end{aligned}$$

Setting

$$A = \frac{2(x+y)^2+1}{\sqrt{xy}} \quad \text{and} \quad B = 2\sqrt{xy},$$

the above inequality as

$$\frac{9(x+y)}{M} \le (A-B)^2.$$

Note that $A > B > 0$ as $A - B = \frac{x^2+y^2+(x+y)^2+1}{\sqrt{xy}} > 0$. For real numbers x and y with fixed $x+y$, if we increasing the value of $\sqrt{xy}$, the left-hand side $(9(x+y)/M)$ of the above inequality does not change its value, while A decrease its value (with fixed numerate and increasing denominator) and B increase it value. Hence, if we increasing the value of $\sqrt{xy}$, $A-B$ is a positive term with decreasing value; that is, the right-hand side of the inequality decreases its value. Therefore, when we increases the value of $\sqrt{xy}$ with fixed $x+y$, the above inequality gets strengthened. Therefore, we may assume that $x=y$ in the above inequality, and (‡) becomes

$$18x^3 \le M(6x^2+1)^2 = M(36x^4+12x^2+1)$$

or

$$36x + \frac{12}{x} + \frac{1}{x^3} \ge \frac{18}{M}.$$

It suffices to find the minimum value of the continues function

$$f(x) = 36x + \frac{12}{x} + \frac{1}{x^3} \quad \text{for } x > 0.$$

Note that

$$\frac{df}{dx} = 36 - \frac{12}{x^2} - \frac{3}{x^4} = \frac{3(2x^2-1)(6x^2+1)}{x^4},$$

implying the only critical value $x = \frac{1}{\sqrt{2}}$ in the domain. It is easy to check that $f(x)$ indeed obtains global minimum $32\sqrt{2}$ at $x = \frac{1}{\sqrt{2}}$ in the domain.

We conclude the minimum value of M is $\frac{9\sqrt{2}}{32}$, obtained when $x = y = a-b = b-c = \frac{1}{\sqrt{2}}$ (and $a+b+c=1$); that is,

$$(a,b,c) = \left(\frac{1}{3} + \frac{1}{\sqrt{2}}, \frac{1}{3}, \frac{1}{3} - \frac{1}{\sqrt{2}}\right).$$

Second Solution: (By Aleksandar Ivanov, observer with the Bulgarian team) By the **AM-GM Inequality**, we have

$$\begin{aligned}
& x^2+y^2+(x+y)^2+1 \\
= & \left(x^2+\frac{1}{2}\right)+\left(y^2+\frac{1}{2}\right)+\frac{(x+y)^2}{2}+\frac{(x+y)^2}{2} \\
\geq & \sqrt{2}x+\sqrt{2}y+2xy+\frac{(x+y)^2}{2} \\
\geq & 4\sqrt[4]{(\sqrt{2}x)(\sqrt{2}y)(2xy)\cdot\frac{(x+y)^2}{2}} \\
= & 4\sqrt[4]{2x^2y^2(x+y)^2},
\end{aligned}$$

or

$$(x^2+y^2+(x+y)^2+1)^2 \geq 16\sqrt{2}xy(x+y).$$

It is then routine to show (‡). Equality holds only if $x^2=y^2=\frac{1}{2}$, and it is straightforward to check it indeed leads to the equality case.

4. Determine all pairs (x,y) of integers such that

$$1+2^x+2^{2x+1}=y^2.$$

Note: The answers are $(x,y)=(0,\pm 2)$ and $(x,y)=(4,\pm 23)$. It is easy to check that these are solutions. If (x,y) is a solution then obviously $x\geq 0$ and $(x,-y)$ is a solution too. For $x=0$, we get the first two solutions. Now we assume that (x,y) is a solution with $x>0$; without loss of generality confine attention to $y>0$.

First Solution: The equation rewritten as

$$2^x(1+2^{x+1})=y^2-1=(y-1)(y+1)$$

shows that $\gcd(y-1,y+1)=2$, and exactly one of them divisible by 4. Hence $x\geq 3$ and one of $y-1$ and $y+1$ is divisible by 2^{x-1} but not by 2^x. Consequently, we may write

$$y=2^{x-1}m+\epsilon, \tag{†}$$

where m is odd and $\epsilon=\pm 1$. Plugging this into the original equation we obtain

$$2^x(1+2^{x+1})=(2^{x-1}m+\epsilon)^2-1=2^{2x-2}m^2+2^x m\epsilon,$$

or

$$1+2^{x+1}=2^{x-2}m^2+m\epsilon.$$

It follows that

$$1-m\epsilon=2^{x-2}(m^2-8). \tag{‡}$$

If $\epsilon=1$, (‡) becomes $m^2-8\leq 0$, or $m=1$, which fails to satisfy (‡). Thus $\epsilon=-1$, so (‡) becomes

$$1+m=2^{x-2}(m^2-8)\geq 2(m^2-8),$$

implying that $2m^2 - m - 17 \leq 0$. Hence $m \leq 3$. On the other hand, $m \neq 1$ by (‡). Because m is odd, $m = 3$, leading to $x = 4$ by (‡). Substituting these into (†) yields $y = 23$, completing our proof.

Second Solution: It is easy to check that there is no solution for $x = 1, 2$, and 3. We assume that (x, y) is a solution with $x \geq 5$ and $y > 0$. Note that

$$\begin{cases} 1 + 2^2 + 2^{2x+1} = y^2 \\ 1 + 2^{x+1} + 2^{2x} = (1 + 2^x)^2. \end{cases}$$

Subtracting the two equations gives

$$[y - (1 + 2^x)][y + (1 + 2^x)] = 2^{2x} - 2^x = 2^x(2^x - 1).$$

It is easy to see that both y and $1 + 2^x$ are odd and that $y > 1 + 2^x$. We must have

$$\begin{cases} y - (1 + 2^x) = 2m \\ y + (1 + 2^x) = 2^{x-1}n, \end{cases} \quad \text{or} \quad \begin{cases} y - (1 + 2^x) = 2^{x-1}n \\ y + (1 + 2^x) = 2m, \end{cases}$$

where m and n are positive integers with $mn = 2^x - 1$. It is not difficult to see that the later case is not possible. (Indeed, $y = 2m - (1 + 2^x) \leq 2(2^x - 1) - (1 + 2^x) = 2^x - 3$, contradicting the fact that $y > 1 + 2^x$.) Hence we must have the former case. Solving the system gives

$$y = m + 2^{x-2}n \quad \text{and} \quad 1 + 2^x = 2^{x-2}n - m. \tag{$*$}$$

We claim that $n = 5$. Note that both m and n are odd. We establish our claim by showing that $3 < n < 7$. Since $y > 1 + 2^x$, we have

$$2^{x+1} + 2 = 2(1 + 2^x) < y + 1 + 2^x = 2^{x-1}n,$$

implying that n is greater than 3. Hence $n \geq 5$. By the second equation in $(*)$, we have $m = 2^{x-2}n - 2^x - 1 \geq 5 \cdot 2^{x-2} - 2^x - 1 = 2^{x-2} - 1$. If $n \geq 7$, then

$$2^x - 1 = mn > (2^{x-2} - 1)7 = 2^x + 3 \cdot 2^{x-2} - 7 > 2^x - 1$$

for $x \geq 3$. We conclude that $3 < n < 7$; that is, $n = 5$.

Substituting $n = 5$ in the second equation, and then the first equation in $(*)$ gives $m = 5 \cdot 2^{x-2} - 1 - 2^x = 2^{x-2} - 1$ and $y = m + 2^{x-2}n = 3 \cdot 2^{x-1} - 1$. It follows that

$$(3 \cdot 2^{x-1} - 1)^2 = y^2 = 1 + 2^x + 2^{2x+1}$$

or $9 \cdot 2^{2x-2} - 3 \cdot 2^x = 2^x + 2^{2x+1}$. Solving the last equation gives $4 \cdot 2^x = 2^{2x-2}$, leading to $x = 4$, contradicting the assumption $x \geq 5$. Hence there is no solution for $x \geq 5$.

5. Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial

$$Q(x) = \underbrace{P(P(\ldots(P(x)\ldots))}_{k\ P\text{'s}}$$

Prove that there are at most n integers t such that $Q(t) = t$.

Solution: Let $\mathbb{N}$ denote the set of integers. We define

$$S_P = \{t \mid t \in \mathbb{N} \text{ and } P(t) = t\} \quad \text{and} \quad S_Q = \{t \mid t \in \mathbb{N} \text{ and } Q(t) = t\}.$$

Clearly, S_P is a subset of S_Q. Also note that there are at most n elements in S_P. This is so because that $t \in S_P$ if and only if t is a root of polynomial $P(x) - x = 0$ of degree n, which has at most n roots. If $S_Q = S_P$, we have nothing to prove. We assume that S_P is a proper subset of S_Q, and that $t \in S_Q$ but $t \notin S_P$.

Consider the sequence $\{t_i\}_{i=0}^{\infty}$ with $t_0 = t$, $t_{i+1} = P(t_i)$ for every nonnegative integer i. Since $t \in S_Q$, $t_k = Q(t_0) = Q(t) = t = t_0$.

Since polynomial $a - b$ divides polynomial $a^m - b^m$ (where m is a nonnegative integer. It is not difficult to see that polynomial $a - b$ divides polynomial $P(a) - P(b)$, where $P(x)$ is a polynomial with integer coefficients. Back to our current problem, we conclude that the integer sequence $\{t_i\}_{i=0}^{\infty}$ satisfies the following sequence of divisibility relations

$$(t_{i+1} - t_i) \mid (P(t_{i+1}) - P(t_i)) = t_{i+2} - t_{i+1}$$

for every nonnegative integer i. Since $t_{k+1} - t_k = t_1 - t_0 = P(t) - t \neq 0$, each term in the chain of differences

$$t_1 - t_0,\ t_2 - t_1, \ldots, t_k - t_{k-1},\ t_{k+1} - t_k$$

is a nonzero divisor of the next one, and since $t_{k+1} - t_k = t_1 - t_0$, all these differences have equal absolute values. Let $t_i = \max\{t_0, t_1, \ldots, t_k\}$. Then $t_{i-1} - t_i = -(t_i - t_{i+1})$, or $t_{i-1} = t_{i+1}$. It is then not difficult to see that $t_{i+2} = t_i$ for every i; that is,

$$t_1 = P(t_0) \quad \text{and} \quad t_0 = P(t_1) \quad \text{or} \quad P(P(t_0)) = t_0.$$

Therefore,

$$S_Q = \{t \mid t \in \mathbb{N} \text{ and } P((P(t)) = t\}.$$

Without loss of generality, we may assume that $t_0 < t_1$. If s_0 is another element in S_Q, let $s_1 = P(s_0)$. (It is possible that $s_0 \in S_P$; that is, $s_1 = s_0$.) We further assume without loss of generality that $s_0 < s_1$ and $t_0 < s_0$; that is, $t_0 < s_0 \leq s_1$ and $t_0 < t_1$. Note that $s_1 - t_0$ divides $P(s_1) - P(t_0) = s_0 - t_1$. We must have $t_0 < s_0 < s_1 < t_1$. Note that $s_0 - t_1$ also divides $P(s_0) - P(t_1) = s_1 - t_0$, it follows that $s_0 - t_1 = -(s_1 - t_0)$; that is,

$$t_0 + t_1 = s_0 + s_1 = s_0 + P(s_0).$$

In other words, s_0 is a root of the polynomial $P(x) + x = t_0 + t_1$. Since $P(x) + x$ has degree n, there are at most n (integer) roots (including t_0) of $P(x) + x$. Hence there are at most n elements in S_Q, completing our proof.

6. Assign to each side b of a convex polygon $\mathcal{P}$ the maximum area of a triangle that has b as a side and is contained in $\mathcal{P}$. Show that the sum of the areas assigned to the sides of $\mathcal{P}$ is at least twice the area of $\mathcal{P}$.

Solution: Define the *weight* of a side XY to be the area assigned to it, and define an *antipoint* of a side of a polygon to be one of the points in the polygon farthest from that side (and consequently forming the triangle with greatest area).

Lemma For any side XY, Z is an antipoint if and only if the line l through Z parallel to XY does not go through the interior of the polygon. (Note that this means we can assume Z is a vertex, as we shall do henceforth).

proof: Clearly, if Z is an antipoint l must not go through the interior of the polygon. Now if l does not go through the interior of the polygon, assume there is a point Z' farther away from XY than Z. Since the polygon is convex, the point $XZ' \cap l$ is in the interior of the polygon, which is a contradiction. ■

Suppose for the sake of contradiction that the sum of the weights of the sides is less than twice the area of some polygon. Then let S be the non-empty set of all convex polygons for which the sum of the weights is strictly less than twice the area. It is easy to check that no polygon in S can be a triangle, so we may assume all polygons in S have at least 4 sides.

We first prove by contradiction that there is some polygon in S such that all of its sides are parallel to some other side. Suppose the contrary; then consider one of the polygons in S which has the minimal number of sides not parallel to any other side (this exists by the well-ordering principle). Call this polygon $P = A_1A_2 \cdots A_n$, and WLOG let A_nA_1 be a side which is not parallel to any other side of P.

Then let A_i be the unique antipoint of A_nA_1, and let A_u and A_v be respective antipoints of $A_{i-1}A_i$ and A_iA_{i+1}. Define X to be the point such that $A_uX||A_{i-1}A_i$, $A_vX||A_iA_{i+1}$.

Now consider the set $T \subset P$ of points that are strictly on the same side of A_uA_v as A_nA_1. First of all, for any side in T, A_i must be its antipoint, since the line through A_i parallel to A_jA_{j+1} does not go through the interior of P. Similarly, any vertex in T is not the antipoint of any side.

We now look at the polygon $P' = A_vA_{v+1} \cdots A_{u-1}A_uX$. First of all, it is clear that P' has fewer sides which are not parallel to any other side than P. Using $[\,\cdot\,]$ to denote area, we have

$$[P'] - [P] = [A_1A_2 \cdots A_{v-1}A_vXA_uA_{u+1} \cdots A_n].$$

The weights of the side A_jA_{j+1} is the same in both P' and P for $v \le j < u$, but for P', the sum of the weights of the remaining two sides is $[XA_uA_iA_v]$, as A_i is an antipoint of both A_uX and A_vX. Meanwhile, the sum of the weights of remaining sides for P is $[A_1A_2 \cdots A_{v-1}A_vA_iA_uA_{u+1} \cdots A_n]$. Hence the difference in the sums of weights of P' and P is

$$[XA_uA_iA_v] - [A_1A_2 \cdots A_{v-1}A_vA_iA_uA_{u+1} \cdots A_n] = [A_1A_2 \cdots A_{v-1}A_vXA_uA_{u+1} \cdots A_n],$$

the same as the difference in area (and both differences were positive). Therefore, if the sum of weights of P was less than $2[P]$, then certainly the sum of weights of P' must be less than $2[P']$, so that $P' \in S$. However, this contradicts the minimality of the number of non-parallel sides in P, so there exists a polygon in S with opposite sides parallel.

Now, we will let R be the non-empty set of all polygons in S with all sides parallel to the opposite side. Note that all polygons in R must have an even number of sides. We will show that there is a parallelogram in R.

Suppose not, and that $Q = B_1B_2 \cdots B_{2m}$ is one of the polygons in R with the minimal number of sides, and $m \ge 3$. Let $X = B_1B_2 \cap B_{2m-1}B_{2m}$ and $Y = B_{m-1}B_m \cap B_{m+2}B_{m+1}$. Set $Q' = XB_2B_3 \cdots B_{m-1}YB_{m+2} \cdots B_{2m}$. We propose that the increase in the sum of weights going from Q to Q' is at most twice the increase in area, so that $Q' \in R$.

To aid us, we will let h_X and h_Y be the respective distances of X and Y from $B_{2m}B_1$ and B_mB_{m+1}. The increase in weight is

$$\begin{aligned}&[XB_{m+1}B_1] + [XB_{2m}B_m] + [YB_mB_{2m}] + [YB_{m+1}B_1] - [B_1B_{2m}B_m] - [B_{2m}B_mB_{m+1}]\\ &= [XB_1Y] + [XB_{2m}Y] - [B_1B_{2m}B_m] + [YB_mX] + [YB_{m+1}X] - [B_{2m}B_mB_{m+1}]\\ &= [XB_1B_{2m}] + \frac{h_Y \cdot B_1B_{2m}}{2} + [YB_mB_{m+1}] + \frac{h_X \cdot B_mB_{m+1}}{2}\end{aligned}$$

while the increase in area is $[XB_1B_{2m}] + [YB_mB_{m+1}]$. It remains to show that the first expression is at most twice the second, or in other words, to show that

$$\frac{h_Y \cdot B_1B_{2m}}{2} + \frac{h_X \cdot B_mB_{m+1}}{2} \leq [XB_1B_{2m}] + [YB_mB_{m+1}] = \frac{h_X \cdot B_1B_{2m}}{2} + \frac{h_Y \cdot B_mB_{m+1}}{2},$$

which is equivalent to

$$(h_X - h_Y)(B_1B_{2m} - B_mB_{m+1}) \geq 0$$

Noting that triangles $B_1B_{2m}X$ and $B_{m+1}B_mY$ are similar, we have $h_X/h_Y = B_1B_{2m}/B_mB_{m+1}$, so the above inequality holds.

With the inequality proven, we now know that $Q' \in R$, and yet Q' has fewer sides than Q. This contradicts the minimality of the number of sides of Q, so there exists a parallelogram in R. However, the sum of the weights of a parallelogram clearly equals twice its area, so this contradicts the entire existence of S, as desired.

7 Appendix

7.1 2005 Olympiad Results

The top twelve students on the 2005 USAMO were (in alphabetical order):

Robert Cordwell	Manzano High School	Albuquerque, NM
Zhou Fan	Parsippany Hills High School	Parsippany, NJ
Sherry Gong	Phillips Exeter Academy	Exeter, NH
Rishi Gupta	Henry M. Gunn High School	Palo Alto, CA
Hyun Soo Kim	Academy of Advancement in Science and Tech	Hackensack, NJ
Brian Lawrence	Montgomery Blair High School	Silver Spring, MD
Albert Ni	Illinois Math and Science Academy	Aurora, IL
Natee Pitiwan	Brooks School	North Andover, MA
Eric Price	Thomas Jefferson HS of Science and Tech	Alexandria, VA
Peng Shi	Sir John A. MacDonald Collegiate Institute	Toronto, ON
Yi Sun	The Harker School	San Jose, CA
Yufei Zhao	Don Mills Collegiate Institute	Toronto, ON

Brian Lawrence, was the winner of the Samuel Greitzer-Murray Klamkin award, given to the top scorer(s) on the USAMO. Brian Lawrence and Eric Price placed first and second, respectively, Peng Shi and Yufei Zhao tied for third, on the USAMO. They were awarded college scholarships of \$20000, \$15000, \$5000, and \$5000, respectively, by the Akamai Foundation. The Clay Mathematics Institute (CMI) award, for a solution of outstanding elegance, and carrying a \$5000 cash prize, was presented to Sherry Gong for her solution to USAMO Problem 3.

The USA team members were chosen according to their combined performance on the 34$^{\text{th}}$ annual USAMO, and the Team Selection Test that took place at the Mathematical Olympiad Summer Program (MOSP), held at the University of Nebraska-Lincoln, June 12 - July 2, 2005. Members of the USA team at the 2005 IMO (Mérida, México) were Robert Cordwell, Sherry Gong, Hyun Soo Kim, Brian Lawrence, Thomas Mildorf, and Eric Price. Zuming Feng (Phillips Exeter Academy) and Melanie Wood (Princeton University) served as team leader and deputy leader, respectively. The team was also accompanied by Steven Dunbar (University of Nebraska-Lincoln), as observers of the deputy leader.

There were 513 contestants in the 2005 IMO. The average score is 13.97 (out of 42) points. Gold medals were awarded to students scoring between 35 and 42 points, silver medals to students scoring between 23 and 34 points, and bronze medals to students scoring between 12 and 22 points. There were 42 gold medalists, 79 silver medalists, and 122 bronze medalists. Brian submitted one of the 16 perfect papers. Moldovian contestant Iurie Boreico's elegant solution on problem 3 won a special award in the IMO, the first time this award is given in the past 10 years. from The team's individual performances were as follows:

Cordwell	GOLD Medallist	Gong	SILVER Medallist
Kim	SILVER Medallist	Lawrence	GOLD Medallist
Mildorf	GOLD Medallist	Price	GOLD Medallist

In terms of total score (out of a maximum of 252), the highest ranking of the 93 participating teams were as follows:

China	235	Iran	201	Taiwan	190	Ukraine	181
USA	213	Korea	200	Japan	188	Bulgaria	173
Russia	212	Romania	191	Hungary	181	Germany	163

7.2 2006 Olympiad Results

The top twelve students on the 2006 USAMO were (in alphabetical order):

Yakov Berchenko-Kogan	Needham B. Broughton High School	Raleigh, NC
Yi Han	Phillips Exeter Academy	Exeter, NH
Sherry Gong	Phillips Exeter Academy	Exeter, NH
Taehyeon Ko	Phillips Exeter Academy	Exeter, NH
Brian Lawrence	Montgomery Blair High School	Silver Spring, MD
Tedrick Leung	North Hollywood High School	North Hollywood, CA
Richard McCutchen	Montgomery Blair High School	Silver Spring, MD
Peng Shi	Sir John A. MacDonald C.I.	Toronto, ON
Yi Sun	The Harker School	San Jose, CA
Arnav Tripathy	East Chapel Hill High School	Chapel Hill, NC
Alex Zhai	University Laboratory High School	Urbana, IL
Yufei Zhao	Don Mills Collegiate Institute	Toronto, ON

Brian Lawrence was the winner of the Samuel Greitzer–Murray Klamkin award, given to the top scorer(s) on the USAMO. Brian Lawrence, Alex Zhai, and Yufei Zhao placed first, second, and third, respectively. They were awarded college scholarships of $20000, $15000, $10000, respectively, by the Akamai Foundation. The Clay Mathematics Institute (CMI) award, for a solution of outstanding elegance, and carrying a $5000 cash prize, was presented to Brian Lawrence for his solution to USAMO Problem 5, presented as the third solution here.

The USA team members were chosen according to their combined performance on the 35$^{\text{th}}$ annual USAMO, and the Team Selection Test that took place at the Mathematical Olympiad Summer Program (MOSP), held at the University of Nebraska-Lincoln, June 5 – July 1, 2005. Members of the USA team at the 2006 IMO (Ljubljana, Slovenia) were Zachary Abel, Zarathustra (Zeb) Brady, Taehyeon (Ryan) Ko, Yi Sun, Arnav Tripathy, and Alex Zhai. Zuming Feng (Phillips Exeter Academy) and Alex Saltman (Stanford University) served as team leader and deputy leader, respectively. The team was also accompanied by Steven Dunbar (University of Nebraska-Lincoln), as the observer of the deputy leader.

There were 498 contestants from 90 countries in the 2006 IMO. Gold medals were awarded to students scoring between 28 and 42 points, silver medals to students scoring between 19 and 27 points, and bronze medals to students scoring between 15 and 18 points. There were 42 gold medalists, 89 silver medalists, 122 bronze medalists, and honorable mentions (awarding to non-medalists solving at least one problem completely). There were 3 perfect papers (Iurie Boreico from Republic of Moldova, Zhiyu Liu from People's Republic of China, and Alexander Magazinov from Russian Federation) on this difficult exam, even though it has two relatively easy entry level problems (in problems 1 and 4). Tripathy's 30 tied for 16$^{\text{th}}$ place overall. The team's individual performances were as follows:

Able	SILVER Medallist	Brady	GOLD Medallist
Ko	SILVER Medallist	Sun	SILVER Medallist
Tripathy	GOLD Medallist	Zhai	SILVER Medallist

In terms of total score (out of a maximum of 252), the highest ranking of the 90 participating teams were as follows:

China	214	Germany	157	Japan	146	Taiwan	136
Russia	174	USA	154	Iran	145	Poland	133
Korea	170	Romania	152	Moldova	140	Italy	132

7.3 2007 Olympiad Results

The top twelve students on the 2007 USAMO were (in alphabetical order):

Sergei Bernstein	Belmont High School	Belmont, MA
Sherry Gong	Phillips Exeter Academy	Exeter, NH
Adam Hesterberg	Garfield High School	Seattle, WA
Eric Larson	South Eugene High School	Eugene, OR
Brian Lawrence	Montgomery Blair High School	Kensington, MD
Tedrick Leung	North Hollywood High School	Winnetka, CA
Haitao Mao	Thomas Jefferson HS of Science and Tech	Vienna, VA
Delong Meng	Baton Rouge Magnet High School	Baton Rouge, LA
Krishanu Sankar	Horace Mann High School	Hastings on Hudson, NY
Jacob Steinhardt	Thomas Jefferson HS of Science and Tech	Vienna, VA
Arnav Tripathy	East Chapel Hill High School	Chapel Hill, NC
Alex Zhai	University laboratory High School	Champaign, IL

Brian Lawrence was the winner of the Samuel Greitzer–Murray Klamkin award, given to the top scorer(s) on the USAMO. Sherry Gong and Alex Zhai tied for second place. Brian Lawrence, Sherry Gong, and Alex Zhai were awarded college scholarships of \$20000, \$15000, \$15000, respectively, by the Akamai Foundation. The Clay Mathematics Institute (CMI) award, for a solution of outstanding elegance, and carrying a \$5000 cash prize, was presented to Andrew Geng for his solution to USAMO Problem 4, presented as the second solution here.

The USA team members were chosen according to their combined performance on the 36$^{\text{th}}$ annual USAMO and the Team Selection Test held in Washington, D.C. on May 22 and 23, 2007. Members of the USA team at the 2007 IMO (Hanoi, Vietnam) were Sherry Gong, Eric Larson, Brian Lawrence, Tedrick Leung, Arnav Tripathy, and Alex Zhai. Zuming Feng (Phillips Exeter Academy) and Ian Le served as team leader and deputy leader, respectively. The team was also accompanied by Steven Dunbar (University of Nebraska-Lincoln), as the observer of the deputy leader. The Mathematical Olympiad Summer Program (MOSP) was held at the University of Nebraska-Lincoln, June 10 – June 30, 2007.

For more information about the USAMO or the MOSP, contact Steven Dunbar at `sdunbar@math.unl.edu`.

7.4 2002-2006 Cumulative IMO Results

In terms of total scores (out of a maximum of 1260 points for the last five years), the highest ranking of the participating IMO teams is as follows:

China	1092	Romania	819	Hungary	760	Belarus	647
Russia	962	Vietnam	808	Ukraine	711	Turkey	633
USA	938	Taiwan	791	Germany	706	Poland	605
Bulgaria	877	Japan	780	United Kingdom	654	Hong Kong	598
Korea	856	Iran	779	Canada	648	India	595

More and more countries now value the crucial role of meaningful problem solving in mathematics education. The competition is getting tougher and tougher. A top ten finish is no longer a given for the traditional powerhouses.