USA and International Mathematical Olympiads 2003

Library of Congress Catalog Card Number 2004?????
ISBN 0-88385-???-?
Printed in the United States of America
Current Printing (last digit):
10 9 8 7 6 5 4 3 2 1

USA and International Mathematical Olympiads 2003

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Published and distributed by

The Mathematical Association of America

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Preface

This book is intended to help students preparing to participate in the USA Mathematical Olympiad (USAMO) in the hope of representing the United States at the International Mathematical Olympiad (IMO). The USAMO is the third stage of the selection process leading to participation in the IMO. The preceding examinations are the AMC 10 or the AMC 12 (which replaced the American High School Mathematics Examination) and the American Invitational Mathematics Examination (AIME). Participation in the AIME and the USAMO is by invitation only, based on performance in the preceding exams of the sequence.

The top 12 USAMO students are invited to attend the Mathematical Olympiad Summer Program (MOSP) regardless of their grade in school. Additional MOSP invitations are extended to the most promising non-graduating USAMO students, as potential IMO participants in future years. During the first days of MOSP, IMO-type exams are given to the top 12 USAMO students with the goal of identifying the six members of the USA IMO Team. The Team Selection Test (TST) simulates an actual IMO, consisting of six problems to be solved over two 4 1/2 hour sessions. The 12 equally weighted problems (six on the USAMO and six on the TST) determine the USA Team.

The Mathematical Olympiad booklets have been published since 1976. Copies for each year through 1999 can be ordered from the Mathematical Association of American (MAA) American Mathematics Competitions (AMC). This publication, Mathematical Olympiads 2000, Mathematical Olympiads 2001, and Mathematical Olympiads 2002 are published by the MAA. In addition, various other publications are useful in preparing for the AMC-AIME-USAMO-IMO sequence (see Chapter 6, Further Reading).

For more information about the AMC examinations, or to order Mathematical Olympiad booklets from previous years, please write to

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or visit the AMC web site at `www.unl.edu/amc`.

Acknowledgments

Thanks to Reid Barton, Gregory Galperin, Razvan Gelca, Gerald Heuer, Kiran Kedlaya, Bjorn Poonen, Cecil Rousseau, Alex Saltman, and Zoran Suniḱ for contributing problems to this year's USAMO packet. Special thanks to Reid, Kiran, Bjorn, and Richard Stong for their additional solutions and comments made in their review of the packet. Thanks to Kiran and Richard for their further comments and solutions from grading Problems 3 and 6 on the USAMO. Thanks to Charles Chen, Po-Ru Loh and Tony Zhang who proofread this book. Thanks to Anders Kaseorg, Po-Ru, Tiankai Liu, and Matthew Tang who presented insightful solutions. And, also, thanks to Ian Le, Ricky Liu, and Melanie Wood who took the TST in advance to test the quality of the exam.

Abbreviations and Notations

Abbreviations

IMO	International Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad
MOSP	Mathematical Olympiad Summer Program

Notation for Numerical Sets and Fields

$\mathbb{Z}$	the set of integers
$\mathbb{Z}_n$	the set of integers modulo n

Notations for Sets, Logic, and Geometry

$\iff$	if and only if
$\implies$	implies
$\lvert A\rvert$	the number of elements in set A
$A \subset B$	A is a proper subset of B
$A \subseteq B$	A is a subset of B
$A \setminus B$	A without B (the complement of A with respect to B)
$A \cap B$	the intersection of sets A and B
$A \cup B$	the union of sets A and B
$a \in A$	the element a belongs to the set A
AB	the length of segment AB
$\widehat{AB}$	the arc AB
$\overrightarrow{AB}$	the vector AB

Introduction

Olympiad-style exams consist of several challenging essay-type problems. Correct and complete solutions often require deep analysis and careful argument. Olympiad questions can seem impenetrable to the novice, yet most can be solved by using elementary high school mathematics, cleverly applied.

Here is some advice for students who attempt the problems that follow:

- Take your time! Very few contestants can solve all of the given problems within the time limit. Ignore the time limit if you wish.
- Try the "easier" questions first (problems 1 and 4 on each exam).
- Olympiad problems don't "crack" immediately. Be patient. Try different approaches. Experiment with simple cases. In some cases, working backward from the desired result is helpful.
- If you get stumped, glance at the *Hints* section. Sometimes a problem requires an unusual idea or an exotic technique that might be explained in this section.
- Even if you can solve a problem, read the hints and solutions. They may contain some ideas that did not occur in your solution, and may discuss strategic and tactical approaches that can be used elsewhere.
- The formal solutions are models of elegant presentation that you should emulate, but they often obscure the torturous process of investigation, false starts, inspiration and attention to detail that led to them. When you read the formal solutions, try to reconstruct the thinking that went into them. Ask yourself "What were the key ideas?" "How can I apply these ideas further?"
- Many of the problems are presented together with a collection of remarkable solutions developed by the examination committees, con-

testants, and experts, during or after the contests. For each problem with multiple solutions, some common crucial results are presented at the beginning of these solutions. You are encouraged to either try to prove those results on your own or to independently complete the solution to the problem based on these given results.

- Go back to the original problem later and see if you can solve it in a different way.
- All terms in boldface are defined in the *Glossary*. Use the glossary and the reading list to further your mathematical education.
- Meaningful problem solving takes practice. Don't get discouraged if you have trouble at first. For additional practice, use prior years' exams or the books on the reading list.

I

The Problems

1 USAMO

32nd United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 29, 2003

1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.
2. A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.
3. Let $n \neq 0$. For every sequence of integers

$$a = a_0, a_1, a_2, \dots, a_n$$

satisfying $0 \leq a_i \leq i$, for $i = 0, \dots, n$, define another sequence

$$t(a) = t(a)_0, t(a)_1, t(a)_2, \dots, t(a)_n$$

by setting $t(a)_i$ to be the number of terms in the sequence a that precede the term a_i and are different from a_i. Show that, starting from any sequence a as above, fewer than n applications of the transformation t lead to a sequence b such that $t(b) = b$.

32nd United States of America Mathematical Olympiad

Day II 12:30 PM – 5:00 PM EDT

April 30, 2003

4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Rays BA and ED intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

2 Team Selection Test

44th IMO Team Selection Test

Lincoln, Nebraska

Day I 1:00 PM – 5:30 PM

June 20, 2003

1. For a pair of integers a and b, with $0 < a < b < 1000$, the set $S \subseteq \{1, 2, \ldots, 2003\}$ is called a *skipping set* for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \notin \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b). Determine the maximum and minimum values of f.

2. Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, and PC intersect sides BC, CA, and AB at D, E, and F, respectively. Prove that
$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$
if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

3. Find all ordered triples of primes (p, q, r) such that
$$p \mid q^r + 1, \quad q \mid r^p + 1, \quad r \mid p^q + 1.$$

44th IMO Team Selection Test

Lincoln, Nebraska

Day II 8:30 AM – 1:00 PM

June 21, 2003

4. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(m+n)f(m-n) = f(m^2)$$
for all $m, n \in \mathbb{N}$.

5. Let a, b, c be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that
$$\begin{aligned}&\frac{\sin a \sin(a-b)\sin(a-c)}{\sin(b+c)}\\ +\ &\frac{\sin b \sin(b-c)\sin(b-a)}{\sin(c+a)}\\ +\ &\frac{\sin c \sin(c-a)\sin(c-b)}{\sin(a+b)} \geq 0.\end{aligned}$$

6. Let $\overline{AH_1}, \overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute scalene triangle ABC. The incircle of triangle ABC is tangent to $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at T_1, T_2, and T_3, respectively. For $k = 1, 2, 3$, let P_i be the point on line H_iH_{i+1} (where $H_4 = H_1$) such that $H_iT_iP_i$ is an acute isosceles triangle with $H_iT_i = H_iP_i$. Prove that the circumcircles of triangles $T_1P_1T_2$, $T_2P_2T_3$, $T_3P_3T_1$ pass through a common point.

3 IMO

44th International Mathematical Olympiad

Tokyo, Japan

Day I 9 AM – 1:30 PM

July 13, 2003

1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\} \quad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

2. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

3. A convex hexagon is given in which any two opposite sides have the following property: the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

(A convex $ABCDEF$ has three pairs of opposite sides: AB and DE, BC and EF, CD and FA.)

44th International Mathematical Olympiad

Tokyo, Japan

Day II 9 AM – 1:30 PM

July 14, 2003

4. Let $ABCD$ be a convex quadrilateral. Let P, Q and R be the feet of perpendiculars from D to lines BC, CA and AB, respectively. Show that $PQ = QR$ if and only if the bisectors of angles ABC and ADC meet on segment AC.

5. Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be real numbers with $x_1 \le x_2 \le \cdots \le x_n$.

   (a) Prove that

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$

   (b) Show that the equality holds if and only if $x_1, x_2, \ldots, x_n$ form an arithmetic sequence.

6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

2
Hints

1 USAMO

1. Try small cases, and build up this number one digit a time. This number is unique. We did not ask for the uniqueness in the statement in order to not hint to the approach in the Third Solution.
2. Reduce the problem to a quadrilateral with lots of angles with rational cosine values.
3. If the value of a term stays the same for one step, it becomes stable.
4. Let XYZ be a triangle with M the midpoint of side YZ. Point P lies on segment XM. Lines YP and XZ meet at Q, ZP and XY at R. Then $RQ \parallel YZ$.
5. It suffices to prove the desired result by assuming, additionally, that $a + b + c = 3$.
6. Controlling the maximum of the six numbers is not enough.

2 Team Selection Test

1. The extremes can be obtained by different approaches. One requires the greedy algorithm, another applies congruence theory.
2. Apply the ingredients that prove **Ceva's Theorem** to convert this into an algebra problem.
3. Prove that one of the primes is 2.
4. Play with the given relation and compute many values of the function.
5. Reduce this to *Schur's Inequality*.
6. The common point is the orthocenter of triangle $T_1T_2T_3$.

3 IMO

1. The greedy algorithm works!
2. Assume that $a^2/(2ab^2-b^3+1)=k$, or, $a^2=2ab^2k-b^3k+k$, where k is a positive integer. Consider the quadratic equation $x^2-2b^2kx+(b^3-1)k=0$ for fixed positive integers b and k.
3. View the given conditions as equality cases of some geometric inequalities and consider the angles formed by three major diagonals.
4. Apply the **Extended Law of Sines**.
5. This is a clear-cut application of **Cauchy–Schwarz Inequality**.
6. The prime q is a divisor of p^p-1.

3

Formal Solutions

1 USAMO

1. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

First Solution. We proceed by induction. The property is clearly true for $n = 1$. Assume that $N = a_1a_2 \ldots a_n$ is divisible by 5^n and has only odd digits. Consider the numbers

$$N_1 = 1a_1a_2 \ldots a_n = 1 \cdot 10^n + 5^n M = 5^n(1 \cdot 2^n + M),$$
$$N_2 = 3a_1a_2 \ldots a_n = 3 \cdot 10^n + 5^n M = 5^n(3 \cdot 2^n + M),$$
$$N_3 = 5a_1a_2 \ldots a_n = 5 \cdot 10^n + 5^n M = 5^n(5 \cdot 2^n + M),$$
$$N_4 = 7a_1a_2 \ldots a_n = 7 \cdot 10^n + 5^n M = 5^n(7 \cdot 2^n + M),$$
$$N_5 = 9a_1a_2 \ldots a_n = 9 \cdot 10^n + 5^n M = 5^n(9 \cdot 2^n + M).$$

The numbers $1 \cdot 2^n + M, 3 \cdot 2^n + M, 5 \cdot 2^n + M, 7 \cdot 2^n + M, 9 \cdot 2^n + M$ give distinct remainders when divided by 5. Otherwise the difference of some two of them would be a multiple of 5, which is impossible, because neither 2^n is a multiple of 5, nor is the difference of any two of the numbers $1, 3, 5, 7, 9$. It follows that one of the numbers N_1, N_2, N_3, N_4, N_5 is divisible by $5^n \cdot 5$, and the induction is complete.

Second Solution. For an m digit number a, where $m \geq n$, let $\ell(a)$ denote the $m - n$ leftmost digits of a. (That is, we consider $\ell(a)$ as an $(m - n)$-digit number.) It is clear that we can choose a large odd number k such that $a_0 = 5^n \cdot k$ has at least n digits. Assume that a_0 has m_0 digits, where $m_0 \geq n$. Note that the a_0 is an odd multiple of 5. Hence the unit digit of a_0 is 5.

If the n rightmost digits of a_0 are all odd, then the number $b_0 = a_0 - \ell(a_0) \cdot 10^n$ satisfies the conditions of the problem, because b_0 has only odd digits (the same as those n leftmost digits of a_0) and that b_0 is the difference of two multiples of 5^n.

If there is an even digit among the n rightmost digits of a_0, assume that i_1 is the smallest positive integer such that the i_1th rightmost digit of a_0 is even. Then number $a_1 = a_0 + 5^n \cdot 10^{i_1 - 1}$ is a multiple of 5^n with at least n digits. The $(i-1)$th rightmost digit is the same as that of a_0 and the i_1th rightmost digit of a_1 is odd. If the n rightmost digits of a_1 are all odd, then $b_1 = a_1 - \ell(a_1) \cdot 10^n$ satisfies the conditions of the problem. If there is an even digit among the n rightmost digits of a_1, assume that i_2 is the smallest positive integer such that the i_2th rightmost digit of a_1 is even. Then $i_2 > i_1$. Set $a_2 = a_1 + 5^n \cdot 10^{i_2 - 1}$. We can repeat the above process of checking the rightmost digits of a_2 and eliminate the rightmost even digits of a_2, if there is such a digit among the n rightmost digits of a_2. This process can be repeated for at most $n-1$ times because the unit digit of a_0 is 5. Thus, we can obtain a number a_k, for some nonnegative integer k, such that a_k is a multiple of 5^n with its n rightmost digits all odd. Then $b_k = a_k - \ell(a_k) \cdot 10^n$ is a number that satisfies the conditions of the problem.

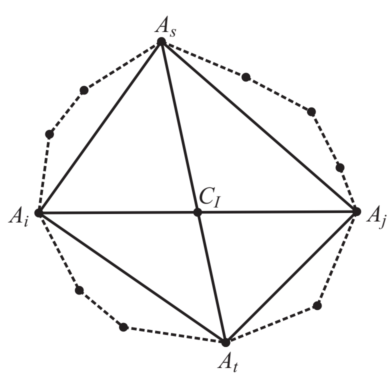
Third Solution. Consider all the nonnegative multiples of 5^n that have no more than n digits. There are 2^n such multiples, namely, $m_0 = 0, m_1 = 5^n, m_2 = 2 \cdot 5^n, \ldots, m_{2^n - 1} = (2^n - 1)5^n$. For each m_i, we define an n-digit binary string $s(m_i)$. If m_i is a k_i-digit number, the leftmost $n - k_i$ digits of $s(m_i)$ are all 0's, and the jth digit, $1 \leq j \leq k_i$, of $s(m_i)$ is 1 (or 0) if the jth rightmost digit of m_i is odd (or even). (For example, for $n = 4$, $m_0 = 0$, $m_1 = 625$, $m_2 = 1250$, and $s(m_0) = 0000$, $s(m_1) = 0001$, $s(m_2) = 1010$.) There are 2^n n-digit binary strings. It suffices to show that s is one-to-one, that is $s(m_i) \neq s(m_j)$ for $i \neq j$. Because then there must be a m_i with $s(m_i)$ being a string of n 1's, that is, m_i has n digits and all of them are odd.

We write m_i and m_j in binary system. Then there is a smallest positive integer k such that the kth rightmost digit in the binary representations of m_i and m_j are different. Without loss of generality, we assume that those kth digits for m_i and m_j are 1 and 0, respectively. Then $m_i = s_i + 2^k + t$ and $m_j = s_j + t$, where s_i, s_j, t are positive integers such that 2^{k+1} divides both s_i and s_j and that $0 \le t \le 2^k - 1$. Note that adding $2^k \cdot 5^n$ to $t \cdot 5^n$ will change the parity of the $(k+1)$th rightmost digit of $t \cdot 5^n$ while not affect the k rightmost digits of $t \cdot 5^n$. Note also that adding $s_i \cdot 5^n$ (or $s_j \cdot 5^n$) to $(2^k + t) \cdot 5^n$ (or $t \cdot 5^n$) will not affect the last $k+1$ digits of $(2^k + t) \cdot 5^n$ (or $t \cdot 5^n$). Hence the $(k+1)$th rightmost digits in the decimal representations of $m_i \cdot 5^n$ and $m_j \cdot 5^n$ have different parities. Thus $s(m_i) \ne s(m_j)$, as desired.

2. A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Solution. Let $\mathcal{P} = A_1A_2\ldots A_n$, where n is an integer with $n \ge 3$. The problem is trivial for $n = 3$ because there are no diagonals and thus no dissections. We assume that $n \ge 4$. Our proof is based on the following Lemma.

Lemma *Let $ABCD$ be a convex quadrilateral such that all its sides and diagonals have rational lengths. If segments AC and BD meet at P, then segments AP, BP, CP, DP all have rational lengths.*



It is clear by the Lemma that the desired result holds when $\mathcal{P}$ is a convex quadrilateral. Let A_iA_j $(1 \le i < j \le n)$ be a diagonal of $\mathcal{P}$. Assume that C_1, C_2, $\ldots$, C_m are the consecutive division points on diagonal A_iA_j (where point C_1 is the closest to vertex A_i and C_m is the closest to A_j). Then the segments $C_\ell C_{\ell+1}$, $1 \le \ell \le m-1$, are the sides of all polygons in the dissection. Let C_ℓ be the point where diagonal A_iA_j meets diagonal A_sA_t. Then quadrilateral $A_iA_sA_jA_t$ satisfies the conditions of the Lemma. Consequently, segments A_iC_ℓ and $C_\ell A_j$ have rational lengths. Therefore, segments $A_iC_1, A_iC_2, \ldots, A_jC_m$ all have rational lengths. Thus, $C_\ell C_{\ell+1} = AC_{\ell+1} - AC_\ell$ is rational. Because i, j, ℓ are arbitrarily chosen, we proved that all sides of all polygons in the dissection are also rational numbers.

Now we present two proofs of the Lemma to finish our proof.

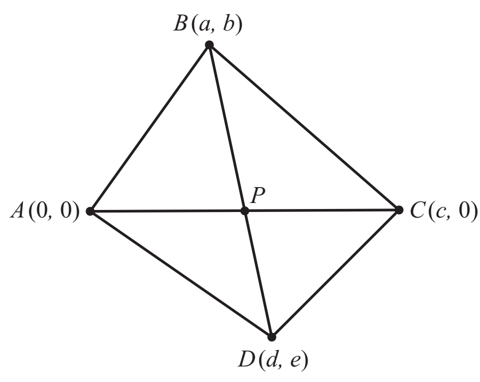
- *First approach* We show only that segment AP is rational, the proof for the others being similar. Introduce Cartesian coordinates with $A = (0, 0)$ and $C = (c, 0)$. Put $B = (a, b)$ and $D = (d, e)$. Then by hypothesis, the numbers

$$AB = \sqrt{a^2 + b^2}, \quad AC = c, \quad AD = \sqrt{d^2 + e^2},$$
$$BC = \sqrt{(a-c)^2 + b^2}, \; BD = \sqrt{(a-d)^2 + (b-e)^2},$$
$$CD = \sqrt{(d-c)^2 + e^2},$$

are rational. In particular,

$$BC^2 - AB^2 - AC^2 = (a-c)^2 + b^2 - (a^2 + b^2) - c^2 = -2ac$$

is rational. Because $c \neq 0$, a is rational. Likewise, d is rational.



Now we have that $b^2 = AB^2 - a^2$, $e^2 = AD^2 - d^2$, and $(b-e)^2 = BD^2 - (a-d)^2$ are rational, and so that $2be = b^2 + e^2 - (b-e)^2$ is rational. Because quadrilateral $ABCD$ is convex, b and e are nonzero and have opposite sign. Hence $b/e = 2be/2b^2$ is rational.

We now calculate

$$P = \left(\frac{bd - ae}{b - e}, 0\right),$$

so

$$AP = \frac{\frac{b}{e} \cdot d - a}{\frac{b}{e} - 1}$$

is rational. ■

- *Second approach* To prove the Lemma, we set $\angle DAP = A_1$ and $\angle BAP = A_2$. Applying the **Law of Cosines** to triangles ADC, ABC, ABD shows that angles A_1, A_2, A_1+A_2 all have rational cosine values. By the Addition formula, we have

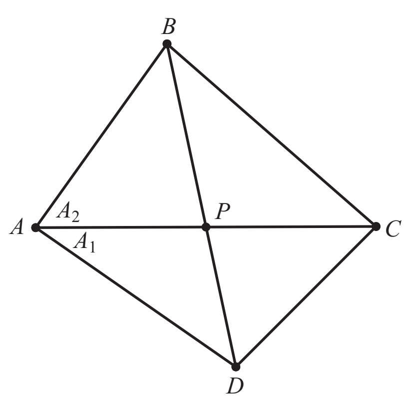
$$\sin A_1 \sin A_2 = \cos A_1 \cos A_2 - \cos(A_1 + A_2),$$

implying that $\sin A_1 \sin A_2$ is rational.

Thus

$$\frac{\sin A_2}{\sin A_1} = \frac{\sin A_2 \sin A_1}{\sin^2 A_1} = \frac{\sin A_2 \sin A_1}{1 - \cos^2 A_1}$$

is rational.



Note that the ratio between the areas of triangles ADP and ABP is equal to $\frac{PD}{BP}$. Therefore

$$\frac{BP}{PD} = \frac{[ABP]}{[ADP]} = \frac{\frac{1}{2}AB \cdot AP \cdot \sin A_2}{\frac{1}{2}AD \cdot AP \cdot \sin A_1} = \frac{AB}{AD} \cdot \frac{\sin A_2}{\sin A_1},$$

implying that $\frac{PD}{BP}$ is rational. Because $BP + PD = BD$ is rational, both BP and PD are rational. Similarly, AP and PC are rational, proving the Lemma.

3. Let $n \neq 0$. For every sequence of integers

$$a = a_0, a_1, a_2, \ldots, a_n$$

satisfying $0 \le a_i \le i$, for $i = 0, \ldots, n$, define another sequence

$$t(a) = t(a)_0, t(a)_1, t(a)_2, \ldots, t(a)_n$$

by setting $t(a)_i$ to be the number of terms in the sequence a that precede the term a_i and are different from a_i. Show that, starting from any sequence a as above, fewer than n applications of the transformation t lead to a sequence b such that $t(b) = b$.

First Solution. Note first that the transformed sequence $t(a)$ also satisfies the inequalities $0 \le t(a)_i \le i$, for $i = 0, \ldots, n$. Call any integer sequence that satisfies these inequalities an *index bounded sequence*.

We prove now that that $a_i \le t(a)_i$, for $i = 0, \ldots, n$. Indeed, this is clear if $a_i = 0$. Otherwise, let $x = a_i > 0$ and $y = t(a)_i$. None of the first x consecutive terms $a_0, a_1, \ldots, a_{x-1}$ is greater than $x - 1$, so they are all different from x and precede x (see the diagram below). Thus $y \ge x$, that is, $a_i \le t(a)_i$, for $i = 0, \ldots, n$.

	0	1	$\ldots$	$x-1$	$\ldots$	i
a	a_0	a_1	$\ldots$	a_{x-1}	$\ldots$	x
$t(a)$	$t(a)_0$	$t(a)_1$	$\ldots$	$t(a)_{x-1}$	$\ldots$	y

This already shows that the sequences stabilize after finitely many applications of the transformation t, because the value of the index i term in index bounded sequences cannot exceed i. Next we prove that if $a_i = t(a)_i$, for some $i = 0, \ldots, n$, then no further applications of t will ever change the index i term. We consider two cases.

- In this case, we assume that $a_i = t(a)_i = 0$. This means that no term on the left of a_i is different from 0, that is, they are all 0. Therefore the first i terms in $t(a)$ will also be 0 and this repeats (see the diagram below).

	0	1	$\ldots$	i
a	0	0	$\ldots$	0
$t(a)$	0	0	$\ldots$	0

- In this case, we assume that $a_i = t(a)_i = x > 0$. The first x terms are all different from x. Because $t(a)_i = x$, the terms $a_x, a_{x+1}, \ldots, a_{i-1}$ must then all be equal to x. Consequently, $t(a)_j = x$ for $j = x, \ldots, i-1$ and further applications of t cannot change the index i term (see the diagram below).

	0	1	$\ldots$	$x-1$	x	$x+1$	$\ldots$	i
a	a_0	a_1	$\ldots$	a_{x-1}	x	x	$\ldots$	x
$t(a)$	$t(a)_0$	$t(a)_1$	$\ldots$	$t(a)_{x-1}$	x	x	$\ldots$	x

For $0 \leq i \leq n$, the index i entry satisfies the following properties: (i) it takes integer values; (ii) it is bounded above by i; (iii) its value does not decrease under transformation t; and (iv) once it stabilizes under transformation t, it never changes again. This shows that no more than n applications of t lead to a sequence that is stable under the transformation t.

Finally, we need to show that no more than $n - 1$ applications of t is needed to obtain a fixed sequence from an initial $n+1$-term index bounded sequence $a = (a_0, a_1, \ldots, a_n)$. We induct on n.

For $n = 1$, the two possible index bounded sequences $(a_0, a_1) = (0, 0)$ and $(a_0, a_1) = (0, 1)$ are already fixed by t so we need zero applications of t.

Assume that any index bounded sequence $(a_0, a_1, \ldots, a_n)$ reach a fixed sequence after no more than $n - 1$ applications of t. Consider an index bounded sequence $a = (a_0, a_1, \ldots, a_{n+1})$. It suffices to show that a will be stabilized in no more than n applications of t. We approach indirectly by assuming on the contrary that $n + 1$ applications of transformations are needed. This can happen only if $a_{n+1} = 0$ and each application of t increased the index $n + 1$ term by exactly 1. Under transformation t, the resulting value of index i term will not be effected by index j term for $i < j$. Hence by the induction hypothesis, the subsequence $a' = (a_0, a_1, \ldots, a_n)$ will be stabilized in no more than $n-1$ applications of t. Because index n term is stabilized at value $x \leq n$ after no more than $\min\{x, n-1\}$ applications of t and index $n+1$ term obtains value x after exactly x applications of t under our current assumptions. We conclude that the index $n+1$ term would become equal to the index n term after no more than $n - 1$ applications of t. However, once two consecutive terms in a sequence are equal they stay equal and stabilize together. Because the index n term needs no more than $n-1$ transformations to be stabilized, a can be stabilized in no more than $n-1$ applications of t, which contradicts our assumption of $n + 1$ applications needed. Thus our assumption was wrong and we need at most n applications of transformation t to stabilize an $(n+1)$-term index bounded sequence. This completes our inductive proof.

Note. There are two notable variations proving the last step.

- *First variation* The key case to rule out is $t^i(a)_n = i$ for $i = 0, \dots, n$. If $a_n = 0$ and $t(a)_n = 1$, then a has only one nonzero term. If it is a_1, then $t(a) = 0, 1, 1, \dots, 1$ and $t(t(a)) = t(a)$, so $t(t(a))_n \neq 2$; if it is a_i for $i > 1$, then $t(a) = 0, \dots, 0, i, 1, \dots, 1$ and $t(t(a)) = 0, \dots, 0, i, i+1, \dots, i+1$ and $t(t(a))_n \neq 2$. That's a contradiction either way. (Actually we didn't need to check the first case separately except for $n = 2$; if $a_n = a_{n-1} = 0$, they stay together and so get fixed at the same step.)
- *Second variation* Let b_{n-1} be the terminal value of a_{n-1}. Then a_{n-1} gets there at least as soon as a_n does (since a_n only rises one each time, whereas a_{n-1} rises by at least one until reaching b_{n-1} and then stops, and furthermore $a_{n-1} \geq 0 = a_n$ to begin with), and when a_n does reach that point, it is equal to a_{n-1}. (Kiran Kedlaya, one of the graders of this problem, likes to call this a "tortoise and hare" argument—the hare a_{n-1} gets a head start but gets lazy and stops, so the tortoise a_n will catch him eventually.)

Second Solution. We prove that for $n \geq 2$, the claim holds without the initial condition $0 \leq a_i \leq i$. (Of course this does not prove anything stronger, but it's convenient.) We do this by induction on n, the case $n = 2$ being easy to check by hand as in the first solution.

Note that if $c = (c_0, \dots, c_n)$ is a sequence in the image of t, and d is the sequence $(c_1, \dots, c_n)$, then the following two statements are true:

(a) If e is the sequence obtained from d by subtracting 1 from each nonzero term, then $t(d) = t(e)$. (If there are no zero terms in d, then subtracting 1 clearly has no effect. If there is a zero term in d, it must occur at the beginning, and then every nonzero term is at least 2.)

(b) One can compute $t(c)$ by applying t to the sequence $c_1, \dots, c_n$, adding 1 to each nonzero term, and putting a zero in front.

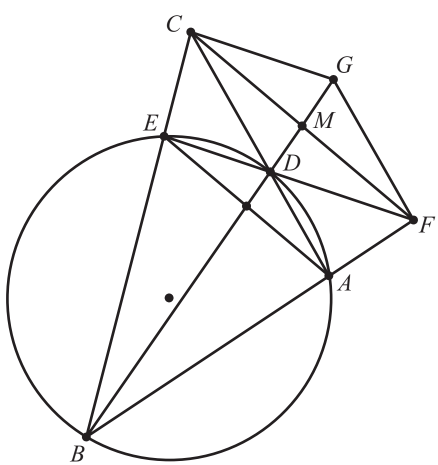
The recipe of (b) works for computing $t^i(c)$ for any i, by (a) and induction on i.

We now apply the induction hypothesis to $t(a)_1, \dots, t(a)_n$ to see that it stabilizes after $n-2$ more applications of t; by the recipe above, that means a stabilizes after $n-1$ applications of t.

Note. A variation of the above approach is the following. Instead of pulling off one zero, pull off *all* initial zeroes of $a_0, \dots, a_n$. (Or rather, pull off all terms equal to the initial term, whatever it is.) Say there are $k+1$ of them (clearly $k \leq n$); after $\min\{k, 2\}$ applications of t, there will be $k+1$ initial zeroes and all remaining terms are at least k. So now $\max\{1, n-k-2\}$ applications of t will straighten out the end, for a total of $\min\{k,2\} + \max\{1, n-k-2\}$. A little case analysis shows that this is good enough: if $k+1 \leq n-1$, then this sum is at most $n-1$ except maybe if $3 > n-1$, i.e., $n \leq 3$, which can be checked by hand. If $k+1 > n-1$ and we assume $n \geq 4$, then $k \geq n-1 \geq 3$, so the sum is $2 + \max\{1, n-k-2\} \leq \max\{3, n-k\} \leq n-1$.

4. Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Rays BA and ED intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

First Solution. Extend segment DM through M to G such that $FG \parallel CD$.



Then $MF = MC$ if and only if quadrilateral $CDFG$ is a parallelogram, or, $FD \parallel CG$. Hence $MC = MF$ if and only if $\angle GCD = \angle FDA$, that is, $\angle FDA + \angle CGF = 180°$.

Because quadrilateral $ABED$ is cyclic, $\angle FDA = \angle ABE$. It follows that $MC = MF$ if and only if

$$180^\circ = \angle FDA + \angle CGF = \angle ABE + \angle CGF,$$

that is, quadrilateral $CBFG$ is cyclic, which is equivalent to

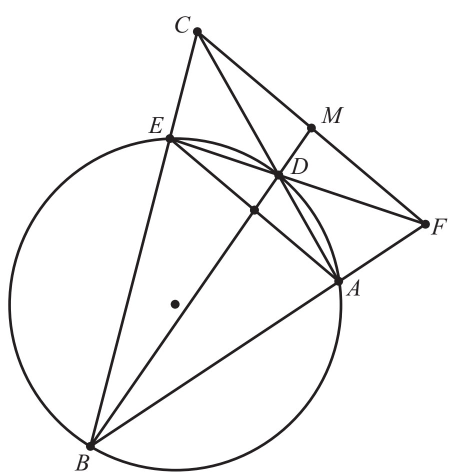
$$\angle CBM = \angle CBG = \angle CFG = \angle DCF = \angle DCM.$$

Because $\angle DMC = \angle CMB$, $\angle CBM = \angle DCM$ if and only if triangles BCM and CDM are similar, that is

$$\frac{CM}{BM} = \frac{DM}{CM},$$

or $MB \cdot MD = MC^2$.

Second Solution.



We first assume that $MB \cdot MD = MC^2$. Because $\frac{MC}{MD} = \frac{MB}{MC}$ and $\angle CMD = \angle BMC$, triangles CMD and BMC are similar. Consequently, $\angle MCD = \angle MBC$. Because quadrilateral $ABED$ is cyclic, $\angle DAE = \angle DBE$. Hence

$$\angle FCA = \angle MCD = \angle MBC = \angle DBE = \angle DAE = \angle CAE,$$

implying that $AE \parallel CF$, and so $\angle AEF = \angle CFE$. Because quadrilateral $ABED$ is cyclic, $\angle ABD = \angle AED$. Hence

$$\angle FBM = \angle ABD = \angle AED = \angle AEF = \angle CFE = \angle MFD.$$

Because $\angle FBM = \angle DFM$ and $\angle FMB = \angle DMF$, triangles BFM and FDM are similar. Consequently, $\frac{FM}{DM} = \frac{BM}{FM}$, or $FM^2 = BM \cdot DM = CM^2$. Therefore $MC^2 = MB \cdot MD$ implies $MC = MF$.

Now we assume that $MC = MF$. Applying **Ceva's Theorem** to triangle BCF and **cevians** BM, CA, FE gives

$$\frac{BA}{AF} \cdot \frac{FM}{MC} \cdot \frac{CE}{EB} = 1,$$

implying that $\frac{BA}{AF} = \frac{BE}{EC}$, so $AE \parallel CF$. Thus, $\angle DCM = \angle DAE$. Because quadrilateral $ABED$ is cyclic, $\angle DAE = \angle DBE$. Hence

$$\angle DCM = \angle DAE = \angle DBE = \angle CBM.$$

Because $\angle CBM = \angle DCM$ and $\angle CMB = \angle DMC$, triangles BCM and CDM are similar. Consequently, $\frac{CM}{DM} = \frac{BM}{CM}$, or $CM^2 = BM \cdot DM$.

Combining the above, we conclude that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

5. Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

First Solution. (Based on work by Matthew Tang and Anders Kaseorg) By multiplying a, b, and c by a suitable factor, we reduce the problem to the case when $a+b+c=3$. The desired inequality reads

$$\frac{(a+3)^2}{2a^2+(3-a)^2} + \frac{(b+3)^2}{2b^2+(3-b)^2} + \frac{(c+3)^2}{2c^2+(3-c)^2} \le 8.$$

Set

$$f(x) = \frac{(x+3)^2}{2x^2+(3-x)^2}$$

It suffices to prove that $f(a)+f(b)+f(c)\le 8$. Note that

$$\begin{aligned} f(x) &= \frac{x^2+6x+9}{3(x^2-2x+3)} = \frac{1}{3}\cdot\frac{x^2+6x+9}{x^2-2x+3} \\ &= \frac{1}{3}\left(1+\frac{8x+6}{x^2-2x+3}\right) \\ &= \frac{1}{3}\left(1+\frac{8x+6}{(x-1)^2+2}\right) \le \frac{1}{3}\left(1+\frac{8x+6}{2}\right) \\ &= \frac{1}{3}(4x+4). \end{aligned}$$

Hence,

$$f(a)+f(b)+f(c) \le \frac{1}{3}(4a+4+4b+4+4c+4) = 8,$$

as desired, with equality if and only if $a=b=c$.

Second Solution. (By Liang Qin) Setting $x=a+b, y=b+c, z=c+a$ gives $2a+b+c=x+z$, hence $2a=x+z-y$ and their analogous forms. The desired inequality becomes

$$\frac{2(x+z)^2}{(x+z-y)^2+2y^2}+\frac{2(z+y)^2}{(z+y-x)^2+2x^2} + \frac{2(y+x)^2}{(y+x-z)^2+2z^2} \le 8.$$

Because $2(s^2+t^2)\ge(s+t)^2$ for all real numbers s and t, we have $2(x+z-y)^2+2y^2 \ge (x+z-y+y)^2=(x+z)^2$. Hence

$$\begin{aligned} \frac{2(x+z)^2}{(x+z-y)^2+2y^2} &= \frac{4(x+z)^2}{2(x+z-y)^2+4y^2} \le \frac{4(x+z)^2}{(x+z)^2+2y^2} \\ &= \frac{4}{1+2\cdot\frac{y^2}{(x+z)^2}} \le \frac{4}{1+2\cdot\frac{y^2}{2(x^2+z^2)}} \\ &= \frac{4(x^2+z^2)}{x^2+y^2+z^2}. \end{aligned}$$

It is not difficult to see that the desired result follows from summing up the above inequality and its analogous forms.

Third Solution. (By Richard Stong) Note that

$$\begin{aligned} (2x+y)^2+2(x-y)^2 &= 4x^2+4xy+y^2+2x^2-4xy+2y^2 \\ &= 3(2x^2+y^2). \end{aligned}$$

Setting $x = a$ and $y = b + c$ yields

$$(2a + b + c)^2 + 2(a - b - c)^2 = 3(2a^2 + (b + c)^2).$$

Thus, we have

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} = \frac{3(2a^2 + (b + c)^2) - 2(a - b - c)^2}{2a^2 + (b + c)^2} = 3 - \frac{2(a - b - c)^2}{2a^2 + (b + c)^2}.$$

and its analogous forms. Thus, the desired inequality is equivalent to

$$\frac{(a - b - c)^2}{2a^2 + (b + c)^2} + \frac{(b - a - c)^2}{2b^2 + (c + a)^2} + \frac{(c - a - b)^2}{2c^2 + (a + b)^2} \ge \frac{1}{2}.$$

Because $(b + c)^2 \le 2(b^2 + c^2)$, we have $2a^2 + (b + c)^2 \le 2(a^2 + b^2 + c^2)$ and its analogous forms. It suffices to show that

$$\frac{(a - b - c)^2}{2(a^2 + b^2 + c^2)} + \frac{(b - a - c)^2}{2(a^2 + b^2 + c^2)} + \frac{(c - a - b)^2}{2(a^2 + b^2 + c^2)} \ge \frac{1}{2},$$

or,

$$(a - b - c)^2 + (b - a - c)^2 + (c - a - b)^2 \ge a^2 + b^2 + c^2.$$

Multiplying this out, the left-hand side of the last inequality becomes $3(a^2 + b^2 + c^2) - 2(ab + bc + ca)$. Therefore the last inequality is equivalent to $2[a^2 + b^2 + c^2 - (ab + bc + ca)] \ge 0$, which is evident because

$$2[a^2 + b^2 + c^2 - (ab + bc + ca)] = (a - b)^2 + (b - c)^2 + (c - a)^2.$$

Equalities hold if and only if $(b+c)^2 = 2(b^2+c^2)$ and $(c+a)^2 = 2(c^2+a^2)$, that is, $a = b = c$.

Fourth Solution. We first convert the inequality into

$$\frac{2a(a + 2b + 2c)}{2a^2 + (b + c)^2} + \frac{2b(b + 2c + 2a)}{2b^2 + (c + a)^2} + \frac{2c(c + 2a + 2b)}{2c^2 + (a + b)^2} \le 5.$$

Splitting the 5 among the three terms yields the equivalent form

$$\sum_{\text{cyc}} \frac{4a^2 - 12a(b + c) + 5(b + c)^2}{3[2a^2 + (b + c)^2]} \ge 0, \tag{1}$$

where $\sum_{\text{cyc}}$ is the **cyclic sum** of variables (a, b, c). The numerator of the term shown factors as $(2a - x)(2a - 5x)$, where $x = b + c$. We will show that

$$\frac{(2a-x)(2a-5x)}{3(2a^2+x^2)} \geq -\frac{4(2a-x)}{3(a+x)}. \qquad (2)$$

Indeed, (2) is equivalent to

$$(2a-x)[(2a-5x)(a+x)+4(2a^2+x^2)] \geq 0,$$

which reduces to

$$(2a-x)(10a^2-3ax-x^2) = (2a-x)^2(5a+x) \geq 0,$$

which is evident. We proved that

$$\frac{4a^2-12a(b+c)+5(b+c)^2}{3[2a^2+(b+c)^2]} \geq -\frac{4(2a-b-c)}{3(a+b+c)},$$

hence (1) follows. Equality holds if and only if $2a = b+c$, $2b = c+a$, $2c = a+b$, i.e., when $a = b = c$.

Fifth Solution. Given a function f of n variables, we define the symmetric sum

$$\sum_{\text{sym}} f(x_1, \ldots, x_n) = \sum_{\sigma} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

where σ runs over all permutations of $1, \ldots, n$ (for a total of $n!$ terms). For example, if $n = 3$, and we write x, y, z for x_1, x_2, x_3,

$$\sum_{\text{sym}} x^3 = 2x^3 + 2y^3 + 2z^3$$

$$\sum_{\text{sym}} x^2y = x^2y + y^2z + z^2x + x^2z + y^2x + z^2y$$

$$\sum_{\text{sym}} xyz = 6xyz.$$

We combine the terms in the desired inequality over a common denominator and use symmetric sum notation to simplify the algebra. The numerator of the difference between the two sides is

$$2\sum_{\text{sym}} 4a^6 + 4a^5b + a^4b^2 + 5a^4bc + 5a^3b^3 - 26a^3b^2c + 7a^2b^2c^2, \qquad (3)$$

and it suffices to show the the expression in (3) is always greater or equal to 0. By the **Weighted AM-GM Inequality**, we have $4a^6 + b^6 + c^6 \geq 6a^4bc$, $3a^5b + 3a^5c + b^5a + c^5a \geq 8a^4bc$, and their analogous forms. Adding those inequalities yields

$$\sum_{\text{sym}} 6a^6 \geq \sum_{\text{sym}} 6a^4bc \quad \text{and} \quad \sum_{\text{sym}} 8a^5b \geq \sum_{\text{sym}} 8a^4bc.$$

Consequently, we obtain

$$\sum_{\text{sym}} 4a^6 + 4a^5b + 5a^4bc \geq \sum_{\text{sym}} 13a^4bc. \tag{4}$$

Again by the AM-GM Inequality, we have $a^4b^2 + b^4c^2 + c^4a^2 \geq 4a^2b^2c^2$, $a^3b^3 + b^3c^3 + c^3a^3 \geq 3a^2b^2c^2$, and their analogous forms. Thus,

$$\sum_{\text{sym}} a^4b^2 + 5a^3b^3 \geq \sum_{\text{sym}} 6a^2b^2c^2,$$

or

$$\sum_{\text{sym}} a^4b^2 + 5a^3b^3 + 7a^2b^2c^2 \geq \sum_{\text{sym}} 13a^2b^2c^2. \tag{5}$$

Recalling **Schur's Inequality**, we have

$$\begin{aligned} &a^3 + b^3 + c^3 + 3abc - (a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2) \\ &\qquad = a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0, \end{aligned}$$

or

$$\sum_{\text{sym}} a^3 - 2a^2b + abc \geq 0.$$

Thus

$$\sum_{\text{sym}} 13a^4bc - 26a^3b^2c + 13a^2b^2c^2 \geq 13abc \sum_{\text{sym}} a^3 - 2a^2b + abc \geq 0. \tag{6}$$

Adding (4), (5), and (6) yields (3).

Note. While the last two methods seem inefficient for this problem, they hold the keys to proving the following inequality:

$$\frac{(b+c-a)^2}{(b+c)^2 + a^2} + \frac{(c+a-b)^2}{(c+a)^2 + b^2} + \frac{(a+b-c)^2}{(a+b)^2 + c^2} \geq \frac{3}{5},$$

where a, b, c are positive real numbers.

6. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Note. Let

$$A \begin{matrix} B & C \\ F & E \end{matrix} D$$

denote a position, where A, B, C, D, E, F denote the numbers written on the vertices of the hexagon. We write

$$A \begin{matrix} B & C \\ F & E \end{matrix} D \pmod 2$$

if we consider the numbers written modulo 2.

This is the hardest problem on the test. Many students thought they had made considerable progress. Indeed, there were only a handful of contestants who were able to find some algorithm without major flaws. Richard Stong, one of the graders of this problem, wrote the following summary.

There is an obvious approach one can take to reducing this problem, namely the greedy algorithm: reducing the largest value. As is often the case, this approach is fundamentally flawed. If the initial values are

$$1 \begin{matrix} 3 & 2 \\ n & 7 \end{matrix} 5$$

where n is an integer greater than 7, then the first move following the greedy algorithm gives

$$1 \begin{matrix} 3 & 2 \\ 6 & 7 \end{matrix} 5.$$

No set of moves can lead from these values to the all zeroes by a parity argument. This example also shows that there is no sequence of moves which always reduces the sum of the six entries and leads to the all zeroes. A correct solution to the problem requires first choosing some parity constraint to avoid the

$$1 \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} 1 \pmod 2$$

situation, which is invariant under the operation. Secondly one needs to find some moves that preserve the chosen constraint and reduce the six values.

Solution. Define the *sum* and *maximum* of a position to be the sum and maximum of the six numbers at the vertices. We will show that from any position in which the sum is odd, it is possible to reach the all-zero position.

Our strategy alternates between two steps:

(a) from a position with odd sum, move to a position with exactly one odd number;

(b) from a position with exactly one odd number, move to a position with odd sum and strictly smaller maximum, or to the all-zero position.

Note that no move will ever increase the maximum, so this strategy is guaranteed to terminate, because each step of type (b) decreases the maximum by at least one, and it can only terminate at the all-zero position. It suffices to show how each step can be carried out.

First, consider a position

$$A \begin{matrix} B & C \\ F & E \end{matrix} D$$

with odd sum. Then either $A + C + E$ or $B + D + F$ is odd; assume without loss of generality that $A + C + E$ is odd. If exactly one of A, C and E is odd, say A is odd, we can make the sequence of moves

$$1 \begin{matrix} B & 0 \\ F & 0 \end{matrix} D \to 1 \begin{matrix} \mathbf{1} & 0 \\ \mathbf{1} & 0 \end{matrix} \mathbf{0} \to \mathbf{0} \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} 0 \to 0 \begin{matrix} 1 & 0 \\ \mathbf{0} & 0 \end{matrix} 0 \pmod 2,$$

where a letter or number in boldface represents a move at that vertex, and moves that do not affect each other have been written as a single move for brevity. Hence we can reach a position with exactly one odd number. Similarly, if A, C, E are all odd, then the sequence of moves

$$1 \begin{matrix} B & 1 \\ F & 1 \end{matrix} D \to 1 \begin{matrix} \mathbf{0} & 1 \\ \mathbf{0} & 1 \end{matrix} \mathbf{0} \to 1 \begin{matrix} 0 & \mathbf{0} \\ 0 & \mathbf{0} \end{matrix} 0 \pmod 2,$$

brings us to a position with exactly one odd number. Thus we have shown how to carry out step (a).

Now assume that we have a position

$$A \begin{matrix} B & C \\ F & E \end{matrix} D$$

with A odd and all other numbers even. We want to reach a position with smaller maximum. Let M be the maximum. There are two cases, depending on the parity of M.

- In this case, M is even, so one of B, C, D, E, F is the maximum. In particular, $A < M$.

  We claim after making moves at B, C, D, E, and F in that order, the sum is odd and the maximum is less than M. Indeed, the following sequence

$$\begin{aligned} &1\begin{matrix}0&0\\0&0\end{matrix}0 \to 1\begin{matrix}\mathbf{1}&0\\0&0\end{matrix}0 \to 1\begin{matrix}1&\mathbf{1}\\0&0\end{matrix}0 \\ \to\ &1\begin{matrix}1&1\\0&0\end{matrix}\mathbf{1} \to 1\begin{matrix}1&1\\0&\mathbf{1}\end{matrix}1 \to 1\begin{matrix}1&1\\\mathbf{0}&1\end{matrix}1 \pmod 2. \end{aligned}$$

shows how the numbers change in parity with each move. Call this new position

$$A'\begin{matrix}B'&C'\\F'&E'\end{matrix}D'.$$

The sum is odd, since there are five odd numbers. The numbers A', B', C', D', E' are all less than M, since they are odd and M is even, and the maximum can never increase. Also, $F' = |A' - E'| \le \max\{A', E'\} < M$. So the maximum has been decreased.

- In this case, M is odd, so $M = A$ and the other numbers are all less than M.

  If $C > 0$, then we make moves at B, F, A, and F, in that order. The sequence of positions is

$$\begin{aligned} &1\begin{matrix}0&0\\0&0\end{matrix}0 \to 1\begin{matrix}\mathbf{1}&0\\0&0\end{matrix}0 \to 1\begin{matrix}1&0\\\mathbf{1}&0\end{matrix}0 \\ \to\ &\mathbf{0}\begin{matrix}1&0\\1&0\end{matrix}0 \to 0\begin{matrix}1&0\\\mathbf{0}&0\end{matrix}0 \pmod 2. \end{aligned}$$

  Call this new position

$$A'\begin{matrix}B'&C'\\F'&E'\end{matrix}D'.$$

  The sum is odd, since there is exactly one odd number. As before, the only way the maximum could not decrease is if $B' = A$; but this is impossible, since $B' = |A - C| < A$ because $0 < C < M = A$. Hence we have reached a position with odd sum and lower maximum.

  If $E > 0$, then we apply a similar argument, interchanging B with F and C with E.

  If $C = E = 0$, then we can reach the all-zero position by the following sequence of moves:

$$A\begin{matrix}B&0\\F&0\end{matrix}D \to A\begin{matrix}\boldsymbol{A}&0\\\boldsymbol{A}&0\end{matrix}\mathbf{0} \to \mathbf{0}\begin{matrix}A&0\\A&0\end{matrix}0 \to 0\begin{matrix}\mathbf{0}&\mathbf{0}\\\mathbf{0}&0\end{matrix}0.$$

  (Here 0 represents zero, not any even number.)

Hence we have shown how to carry out a step of type (b), proving the desired result. The problem statement follows since 2003 is odd.

Note. Observe that from positions of the form

$$\begin{matrix} & 1 & 1 & \\ 0 & & & 0 \\ & 1 & 1 & \end{matrix} \pmod 2 \qquad \text{or rotations}$$

it is impossible to reach the all-zero position, because a move at any vertex leaves the same value modulo 2. Dividing out the greatest common divisor of the six original numbers does not affect whether we can reach the all-zero position, so we may assume that the numbers in the original position are not all even. Then by a more complete analysis in step (a), one can show from any position not of the above form, it is possible to reach a position with exactly one odd number, and thus the all-zero position. This gives a complete characterization of positions from which it is possible to reach the all-zero position.

There are many ways to carry out the case analysis in this problem; the one used here is fairly economical. The important idea is the formulation of a strategy that decreases the maximum value while avoiding the "bad" positions described above.

Second Solution. (By Richard Stong) We will show that if there is a pair of opposite vertices with odd sum (which of course is true if the sum of all the vertices is odd), then we can reduce to a position of all zeros.

Focus on such a pair $\{a, d\}$ with smallest possible $\max\{a, d\}$. We will show we can always reduce this smallest maximum of a pair of opposite vertices with odd sum or reduce to the all-zero position. Because the smallest maximum takes nonnegative integer values, we must be able to achieve the all-zero position.

To see this assume without loss of generality that $a \geq d$ and consider an arc (a, x, y, d) of the position

$$\begin{matrix} & x & y & \\ a & & & d \\ & * & * & \end{matrix}$$

Consider updating x and y alternately, starting with x. If $\max\{x, y\} > a$, then in at most two updates we reduce $\max\{x, y\}$. Thus, we can repeat this *alternate updating* process and we must eventually reach a point when $\max\{x, y\} \leq a$, and hence this will be true from then on.

Under this alternate updating process, the arc of the hexagon will eventually enter a unique cycle of length four modulo 2 in at most one update. Indeed, we have

$$\begin{matrix} & 0 & 0 & \\ 1 & & & 0 \\ & * & * & \end{matrix} \to \begin{matrix} & \mathbf{1} & 0 & \\ 1 & & & 0 \\ & * & * & \end{matrix} \to \begin{matrix} & 1 & \mathbf{1} & \\ 1 & & & 0 \\ & * & * & \end{matrix} \to \begin{matrix} & \mathbf{0} & 1 & \\ 1 & & & 0 \\ & * & * & \end{matrix} \to \begin{matrix} & 0 & \mathbf{0} & \\ 1 & & & 0 \\ & * & * & \end{matrix} \pmod 2$$

and

$$1 \substack{0\ 0\\ \\ *\ *} 0 \to 1 \substack{0\ \mathbf{0}\\ \\ *\ *} 0 \pmod 2; \quad 1 \substack{1\ 0\\ \\ *\ *} 0 \to 1 \substack{\mathbf{1}\ 0\\ \\ *\ *} 0 \pmod 2$$

$$1 \substack{1\ 1\\ \\ *\ *} 0 \to 1 \substack{1\ \mathbf{1}\\ \\ *\ *} 0 \pmod 2; \quad 1 \substack{0\ 1\\ \\ *\ *} 0 \to 1 \substack{\mathbf{0}\ 1\\ \\ *\ *} 0 \pmod 2,$$

or

$$0 \substack{0\ 1\\ \\ *\ *} 1 \to 0 \substack{\mathbf{1}\ 1\\ \\ *\ *} 1 \to 0 \substack{1\ \mathbf{0}\\ \\ *\ *} 1 \to 0 \substack{\mathbf{0}\ 0\\ \\ *\ *} 1 \to 0 \substack{0\ \mathbf{1}\\ \\ *\ *} 1 \pmod 2$$

and

$$0 \substack{0\ 0\\ \\ *\ *} 1 \to 0 \substack{\mathbf{0}\ 0\\ \\ *\ *} 1 \pmod 2; \quad 0 \substack{0\ 1\\ \\ *\ *} 1 \to 0 \substack{0\ \mathbf{1}\\ \\ *\ *} 1 \pmod 2$$

$$0 \substack{1\ 1\\ \\ *\ *} 1 \to 0 \substack{\mathbf{1}\ 0\\ \\ *\ *} 1 \pmod 2; \quad 0 \substack{1\ 0\\ \\ *\ *} 1 \to 0 \substack{1\ \mathbf{0}\\ \\ *\ *} 1 \pmod 2.$$

Further note that each possible parity for x and y will occur equally often.

Applying this alternate updating process to both arcs (a, b, c, d) and (a, e, f, d) of

$$a \substack{b\ \ c\\ \\ f\ \ e} d,$$

we can make the other four entries be at most a and control their parity. Thus we can create a position

$$a \substack{x_1\ \ x_2\\ \\ x_5\ \ x_4} d$$

with $x_i + x_{i+3}$ $(i = 1, 2)$ odd and $M_i = \max\{x_i, x_{i+3}\} \le a$. In fact, we can have $m = \min\{M_1, M_2\} < a$, as claimed, unless both arcs enter a cycle modulo 2 where the values congruent to a modulo 2 are always exactly a. More precisely, because the sum of x_i and x_{i+3} is odd, one of them is not congruent to a and so has its value strictly less than a. Thus both arcs must pass through the state (a, a, a, d) (modulo 2, this is either $(0, 0, 0, 1)$ or $(1, 1, 1, 0)$) in a cycle of length four. It is easy to check that for this to happen, $d = 0$. Therefore, we can achieve the position

$$a \substack{a\ \ a\\ \\ a\ \ a} 0.$$

From this position, the sequence of moves

$$a \substack{a\ \ a\\ \\ a\ \ a} 0 \to a \substack{\mathbf{0}\ \ a\\ \\ \mathbf{0}\ \ a} 0 \to \mathbf{0} \substack{0\ \ \mathbf{0}\\ \\ 0\ \ \mathbf{0}} 0$$

completes the task.

Third Solution. (By Tiankai Liu) In the beginning, because $A+B+C+D+E+F$ is odd, either $A+C+E$ or $B+D+F$ is odd; assume without loss of generality it is the former. Perform the following steps repeatedly.

a. *In this case we assume that A, C, E are all nonzero.* Suppose without loss of generality that $A \geq C \geq E$. Perform the sequence of moves

$$A \begin{matrix} B & C \\ F & E \end{matrix} D \to A \begin{matrix} (\boldsymbol{A}-\boldsymbol{C}) & C \\ (\boldsymbol{A}-\boldsymbol{E}) & E \end{matrix} (\boldsymbol{C}-\boldsymbol{E})$$

$$\to (\boldsymbol{C}-\boldsymbol{E}) \begin{matrix} (A-C) & C \\ (A-E) & (\boldsymbol{A}-\boldsymbol{C}) \end{matrix} (C-E),$$

which decreases the sum of the numbers in positions A, C, E while keeping that sum odd.

b. *In this case we assume that exactly one among A, C, E is zero.* Assume without loss of generality that $A \geq C > E = 0$. Then, because $A+C+E$ is odd, A must be strictly greater than C. Therefore, $-A < A-2C < A$, and the sequence of moves

$$A \begin{matrix} B & C \\ F & 0 \end{matrix} D \to A \begin{matrix} (\boldsymbol{A}-\boldsymbol{C}) & C \\ \boldsymbol{A} & 0 \end{matrix} \boldsymbol{C}$$

$$\to \boldsymbol{C} \begin{matrix} (A-C) & |\boldsymbol{A}-2\boldsymbol{C}| \\ A & 0 \end{matrix} C,$$

decreases the sum of the numbers in positions A, C, E while keeping that sum odd.

c. *In this case we assume that exactly two among A, C, E are zero.* Assume without loss of generality that $A > C = E = 0$. Then perform the sequence of moves

$$A \begin{matrix} B & 0 \\ F & 0 \end{matrix} D \to A \begin{matrix} \boldsymbol{A} & 0 \\ \boldsymbol{A} & 0 \end{matrix} \boldsymbol{0} \to \boldsymbol{0} \begin{matrix} A & 0 \\ A & 0 \end{matrix} 0 \to 0 \begin{matrix} \boldsymbol{0} & 0 \\ \boldsymbol{0} & 0 \end{matrix} 0.$$

By repeatedly applying step (a) as long as it applies, then doing the same for step (b) if necessary, and finally applying step (c) if necessary,

$$0 \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} 0$$

can eventually be achieved.

2 Team Selection Test

1. For a pair of integers a and b, with $0 < a < b < 1000$, the set $S \subseteq \{1, 2, \ldots, 2003\}$ is called a *skipping set* for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \not\in \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b). Determine the maximum and minimum values of f.

Note. This problem caused unexpected difficulties for students. It requires two ideas: applying the greedy algorithm to obtain the minimum and applying the Pigeonhole Principle on congruence classes to obtain the maximum. Most students were successful in getting one of the two ideas and obtaining one of the extremal values quickly, but then many of them failed to switch to the other idea. In turn, their solutions for the second extremal value were very lengthy and sometimes unsuccessful.

Solution. The maximum and minimum values of f are 1334 and 338, respectively.

(a) First, we will show that the maximum value of f is 1334. The set $S = \{1, 2, \ldots, 667\} \cup \{1336, 1337, \ldots, 2002\}$ is a skipping set for $(a, b) = (667, 668)$, so $f(667, 668) \geq 1334$.

Now we prove that for any $0 < a < b < 1000$, $f(a, b) \leq 1334$. Because $a \neq b$, we can choose $d \in \{a, b\}$ such that $d \neq 668$. We assume first that $d \geq 669$. Then consider the $2003 - d \leq 1334$ sets $\{1, d+1\}, \{2, d+2\}, \ldots, \{2003-d, 2003\}$. Each can contain at most one element of S, so $|S| \leq 1334$.

We assume second that $d \leq 667$ and that $\left\lceil \frac{2003}{a} \right\rceil$ is even, that is, $\left\lceil \frac{2003}{a} \right\rceil = 2k$ for some positive integer k. Then each of the congruence classes of $1, 2, \ldots, 2003$ modulo a contains at most $2k$ elements. Therefore at most k members of each of these congruence classes can belong to S. Consequently,

$$
\begin{aligned}
|S| &\leq ka < \frac{1}{2}\left(\frac{2003}{a} + 1\right) a = \frac{2003 + a}{2} \\
&\leq 1335,
\end{aligned}
$$

implying that $|S| \leq 1334$.

Finally, we assume that $d \leq 667$ and that $\left\lceil \frac{2003}{a} \right\rceil$ is odd, that is, $\left\lceil \frac{2003}{a} \right\rceil = 2k + 1$ for some positive integer k. Then, as before, S can contain at most k elements from each congruence class of

$\{1, 2, \ldots, 2ka\}$ modulo a. Then

$$\begin{aligned}|S| &\le ka + (2003 - 2ka) = 2003 - ka \\ &= 2003 - \left(\frac{\lceil \frac{2003}{a} \rceil - 1}{2}\right) a \\ &\le 2003 - \left(\frac{\frac{2003}{a} - 1}{2}\right) a \\ &= \frac{2003 + a}{2} \le 1335.\end{aligned}$$

The last inequality holds if and only if $a = 667$. But if $a = 667$, then $\frac{2003}{a}$ is not an integer, and so the second inequality is strict. Thus, $|S| \le 1334$. Therefore the maximum value of f is 1334.

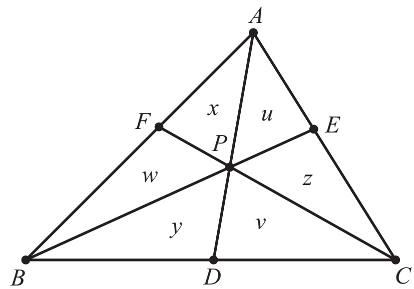
(b) We will now show that the minimum value of f is 668. First, we will show that $f(a, b) \ge 668$ by constructing a skipping set S for any (a, b) with $|S| \ge 668$. Note that if we add x to S, then we are not allowed to add x, $x+a$, or $x+b$ to S at any later time. Then at each step, let us add to S the smallest element of $\{1, 2, \ldots, 2003\}$ that is not already in S and that has not already been disallowed from being in S. Then since adding this element prevents at most three elements from being added at any future time, we can always perform this step $\lceil \frac{2003}{3} \rceil = 668$ times. Thus, $|S| \ge 668$, so $f(a, b) \ge 668$. Now notice that if we let $a = 1, b = 2$, then at most one element from each of the 668 sets $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\ldots$, $\{1999, 2000, 2001\}$, $\{2002, 2003\}$ can belong to S. This implies that $f(1, 2) = 668$, so indeed the minimum value of f is 668.

2. Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, and PC intersect sides BC, CA, and AB at D, E, and F, respectively. Prove that

$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$

if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

Solution. Let $[PAF] = x$, $[PBD] = y$, $[PCE] = z$, $[PAE] = u$, $[PCD] = v$, and $[PBF] = w$.



Note first that

$$\frac{x}{w}=\frac{x+u+z}{w+y+v}=\frac{u+z}{y+v}=\frac{AF}{FB},$$
$$\frac{y}{v}=\frac{x+y+w}{u+v+z}=\frac{x+w}{u+z}=\frac{BD}{DC},$$
$$\frac{z}{u}=\frac{y+z+v}{x+u+w}=\frac{y+v}{x+w}=\frac{CE}{EA}.$$

Point P lies on one of the medians if and only if

$$(x-w)(y-v)(z-u)=0. \qquad (*)$$

By **Ceva's Theorem**, we have

$$\frac{xyz}{uvw}=\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1,$$

or,

$$xyz=uvw. \qquad (1)$$

Multiplying out $\frac{x}{w}=\frac{u+z}{y+v}$ yields $xy+xv=uw+zw$. Likewise, $uy+yz=xv+vw$ and $xz+zw=uy+uv$. Summing up the last three relations, we obtain

$$xy+yz+zx=uv+vw+wu. \qquad (2)$$

Now we are ready to prove the desired result. We first prove the "if" part by assuming that P lies on one of the medians, say AD. Then $y=v$, and so $\frac{y}{v}=\frac{x+w}{u+z}$ and $xyz=uvw$ become $x+w=u+z$ and $xz=uw$, respectively. Then the numbers $x,-z$ and $u,-w$ have the same sum and the same product. It follows that $x=u$ and $z=w$. Therefore $x+y+z=u+v+w$, as desired.

Conversely, we assume that

$$x+y+z=u+v+w. \qquad (3)$$

From (1), (2), and (3) it follows that x, y, z and u, v, w are roots of the same degree three polynomial. Hence $\{x, y, z\} = \{u, v, w\}$. If $x = w$ or $y = v$ or $z = u$, then the conclusion follows by $(*)$. If $x = u$, $y = w$, and $z = v$, then from

$$\frac{x}{w} = \frac{u+z}{y+v} = \frac{u+z-x}{y+v-w} = \frac{z}{v} = 1,$$

we obtain $x = w$. Likewise, we have $y = v$, and so $x = y = z = u = v = w$, that is, P is the centroid of triangle ABC and the conclusion follows. Finally, if $x = v$, $y = u$, $z = w$, then from

$$\frac{x}{w} = \frac{x+u+z}{w+y+v} = \frac{x+y+z}{w+u+v} = 1,$$

we obtain $x = w$. Similarly, $y = v$ and P is again the centroid.

3. Find all ordered triples of primes (p, q, r) such that

$$p \mid q^r + 1, \quad q \mid r^p + 1, \quad r \mid p^q + 1.$$

Solution. Answer: $(2, 5, 3)$ and cyclic permutations.

We check that this is a solution:

$$2 \mid 126 = 5^3 + 1, \quad 5 \mid 10 = 3^2 + 1, \quad 3 \mid 33 = 2^5 + 1.$$

Now let p, q, r be three primes satisfying the given divisibility relations. Since q does not divide $q^r + 1$, $p \neq q$, and similarly $q \neq r$, $r \neq p$, so p, q and r are all distinct. We now prove a lemma.

Lemma. *Let p, q, r be distinct primes with $p \mid q^r + 1$, and $p > 2$. Then either $2r \mid p - 1$ or $p \mid q^2 - 1$.*

Proof. Since $p \mid q^r + 1$, we have

$$q^r \equiv -1 \not\equiv 1 \pmod{p}, \quad \text{because } p > 2,$$

but

$$q^{2r} \equiv (-1)^2 \equiv 1 \pmod{p}.$$

Let d be the order of q mod p; then from the above congruences, d divides $2r$ but not r. Since r is prime, the only possibilities are $d = 2$ or $d = 2r$. If $d = 2r$, then $2r \mid p-1$ because $d \mid p-1$. If $d = 2$, then $q^2 \equiv 1 \pmod{p}$ so $p \mid q^2 - 1$. This proves the lemma. ■

Now let's first consider the case where p, q and r are all odd. Since $p \mid q^r + 1$, by the lemma either $2r \mid p - 1$ or $p \mid q^2 - 1$. But $2r \mid p - 1$ is impossible because

$$2r \mid p-1 \Longrightarrow p \equiv 1 \pmod{r} \Longrightarrow 0 \equiv p^q + 1 \equiv 2 \pmod{r}$$

and $r > 2$. So we must have $p \mid q^2 - 1 = (q-1)(q+1)$. Since p is an odd prime and $q-1$, $q+1$ are both even, we must have

$$p \mid \frac{q-1}{2} \quad \text{or} \quad p \mid \frac{q+1}{2};$$

either way,

$$p \leq \frac{q+1}{2} < q.$$

But then by a similar argument we may conclude $q < r$, $r < p$, a contradiction.

Thus, at least one of p, q, r must equal 2. By a cyclic permutation we may assume that $p = 2$. Now $r \mid 2^q + 1$, so by the lemma, either $2q \mid r - 1$ or $r \mid 2^2 - 1$. But $2q \mid r - 1$ is impossible as before, because q divides $r^2 + 1 = (r^2 - 1) + 2$ and $q > 2$. Hence, we must have $r \mid 2^2 - 1$. We conclude that $r = 3$, and $q \mid r^2 + 1 = 10$. Because $q \neq p$, we must have $q = 5$. Hence (2, 5, 3) and its cyclic permutations are the only solutions.

4. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(m+n)f(m-n) = f(m^2)$$

for all $m, n \in \mathbb{N}$.

Solution. Function $f(n) = 1$, for all $n \in \mathbb{N}$, is the only function satisfying the conditions of the problem.

Note that

$$f(1)f(2n-1) = f(n^2) \quad \text{and} \quad f(3)f(2n-1) = f((n+1)^2)$$

for $n \geq 3$. Thus

$$\frac{f(3)}{f(1)} = \frac{f((n+1)^2)}{f(n^2)}.$$

Setting $\frac{f(3)}{f(1)} = k$ yields $f(n^2) = k^{n-3}f(9)$ for $n \geq 3$. Similarly, for all $h \geq 1$,

$$\frac{f(h+2)}{f(h)} = \frac{f((m+1)^2)}{f(m^2)}$$

for sufficiently large m and is thus also k. Hence $f(2h) = k^{h-1}f(2)$ and $f(2h+1) = k^h f(1)$.

But

$$\frac{f(25)}{f(9)} = \frac{f(25)}{f(23)} \cdots \frac{f(11)}{f(9)} = k^8$$

and

$$\frac{f(25)}{f(9)} = \frac{f(25)}{f(16)} \cdot \frac{f(16)}{f(9)} = k^2,$$

so $k = 1$ and $f(16) = f(9)$. This implies that $f(2h+1) = f(1) = f(2) = f(2j)$ for all j, h, so f is constant. From the original functional equation it is then clear that $f(n) = 1$ for all $n \in \mathbb{N}$.

5. Let a, b, c be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that

$$\frac{\sin a \sin(a-b)\sin(a-c)}{\sin(b+c)} + \frac{\sin b \sin(b-c)\sin(b-a)}{\sin(c+a)} + \frac{\sin c \sin(c-a)\sin(c-b)}{\sin(a+b)} \geq 0.$$

Solution. By the **Product-to-sum formulas** and the **Double-angle formulas**, we have

$$\begin{aligned}\sin(\alpha-\beta)\sin(\alpha+\beta) &= \frac{1}{2}[\cos 2\beta - \cos 2\alpha] \\ &= \sin^2\alpha - \sin^2\beta.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\sin a \sin(a-b)\sin(a-c)\sin(a+b)\sin(a+c) \\ &\quad = \sin c(\sin^2 a - \sin^2 b)(\sin^2 a - \sin^2 c)\end{aligned}$$

and its analogous forms. Therefore, it suffices to prove that

$$x(x^2-y^2)(x^2-z^2) + y(y^2-z^2)(y^2-x^2) + z(z^2-x^2)(z^2-y^2) \geq 0,$$

where $x = \sin a$, $y = \sin b$, and $z = \sin c$ (hence $x, y, z > 0$). Since the last inequality is symmetric with respect to x, y, z, we may assume that $x \geq y \geq z > 0$. It suffices to prove that

$$x(y^2-x^2)(z^2-x^2) + z(z^2-x^2)(z^2-y^2) \geq y(z^2-y^2)(y^2-x^2),$$

which is evident as

$$x(y^2-x^2)(z^2-x^2) \geq 0$$

and

$$z(z^2 - x^2)(z^2 - y^2) \geq z(y^2 - x^2)(z^2 - y^2) \geq y(z^2 - y^2)(y^2 - x^2).$$

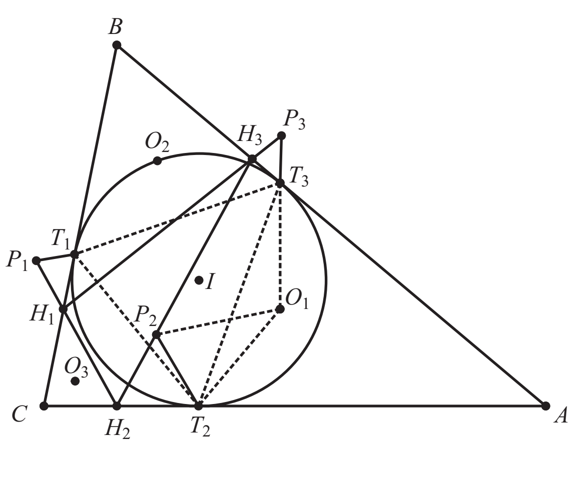
Note. The key step of the proof is an instance of **Schur's Inequality** with $r = \frac{1}{2}$.

6. Let $\overline{AH_1}, \overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute scalene triangle ABC. The incircle of triangle ABC is tangent to $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at T_1, T_2, and T_3, respectively. For $k = 1, 2, 3$, let P_i be the point on line H_iH_{i+1} (where $H_4 = H_1$) such that $H_iT_iP_i$ is an acute isosceles triangle with $H_iT_i = H_iP_i$. Prove that the circumcircles of triangles $T_1P_1T_2$, $T_2P_2T_3$, $T_3P_3T_1$ pass through a common point.

Note. We present three solutions. The first two are synthetic geometry approaches based on the following Lemma. The third solution calculates the exact position of the common point. In these solutions, all angles are directed modulo $180°$. If reader is not familiar with the knowledge of directed angles, please refer our proofs with attached Figures. The proofs of the problem for other configurations can be developed in similar fashions.

Lemma. *The circumcenters of triangles $T_2P_2T_3$, $T_3P_3T_1$, and $T_1P_1T_2$ are the incenters of triangles AH_2H_3, BH_3H_1, and CH_1H_2, respectively.*

Proof. We prove that the circumcenter of triangle $T_2P_2T_3$ is the incenter of triangle AH_2H_3; the other two are analogous. It suffices to show that the perpendicular bisectors of T_2T_3 and T_2P_2 are the interior angle bisectors of $\angle H_3AH_2$ and $\angle AH_2H_3$. For the first pair, notice that

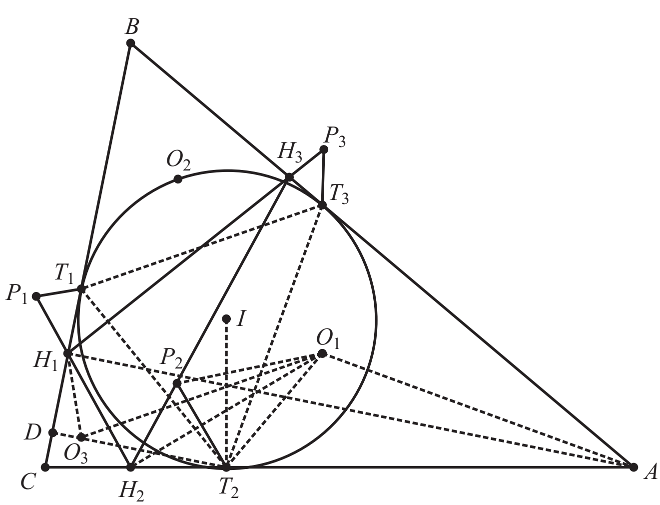


triangle AT_2T_3 is isosceles with $AT_2 = AT_3$ by equal tangents. Also, because triangle ABC is acute, T_2 is on ray AH_2 and T_3 is on ray AH_3. Therefore, the perpendicular bisector of T_2T_3 is the same as the interior angle bisector of $\angle T_3AT_2$, which is the same as the interior angle bisector of $\angle H_3AH_2$.

We prove the second pair similarly. Here, triangle $H_2T_2P_2$ is isosceles with $H_2T_2 = H_2P_2$ by assumption. Also, P_2 is on line H_2H_3 and T_2 is on line H_2A. Because quadrilateral BH_3H_2C is cyclic, $\angle AH_2H_3 = \angle B$ is acute. Now, $\angle T_2H_2P_2$ is also acute by assumption, so P_2 is on ray H_2H_3 if and only if T_2 is on ray H_2A. In other words, $\angle T_2H_2P_2$ either coincides with $\angle AH_2H_3$ or is the vertical angle opposite it. In either case, we see that the perpendicular bisector of T_2P_2 is the same as the interior angle bisector of $\angle T_2H_2P_2$, which is the same as the interior angle bisector of $\angle AH_2H_3$. ■

Let $\omega_1, \omega_2, \omega_3$ denote the circumcircles of triangles $T_2P_2T_3$, $T_3P_3T_1$, $T_1P_1T_2$, respectively. For $i = 1, 2, 3$, let O_i be center of ω_i. By the Lemma, O_1, O_2, O_3 are the incenters of triangles AH_2H_3, BH_3H_1, CH_1H_2, respectively. Let I, ω, and r be the incenter, incircle, and inradius of triangle ABC, respectively.

First Solution. (By Po-Ru Loh) We begin by showing that points O_3, H_2, T_2, and O_3 lie on a cyclic. We will prove this by establishing $\angle O_3O_1H_2 = \angle O_3T_2C = \angle O_3T_2H_2$. To find $\angle O_3O_1H_2$, observe that triangles H_2AH_3 and H_2H_1C are similar. Indeed, quadrilateral BH_3H_2C



is cyclic so $\angle H_2H_3A = \angle C$, and likewise $\angle CH_1H_2 = \angle A$. Now, O_1 and O_3 are corresponding incenters of similar triangles, so it follows that triangles H_2AO_1 and $H_2H_1O_3$ are also similar, and hence are related by a **spiral similarity** about H_2. Thus,

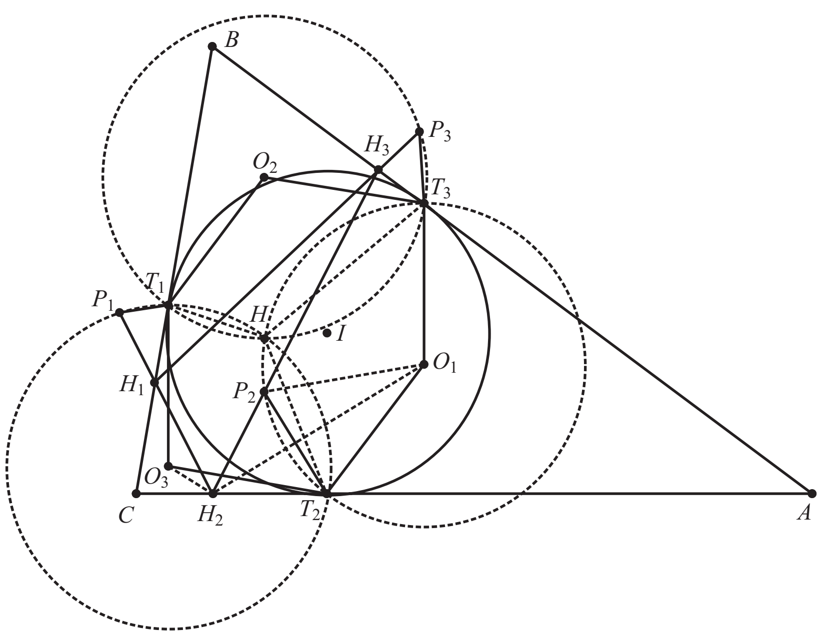
$$\frac{AH_2}{H_1H_2} = \frac{O_1H_2}{O_3H_2}$$

and

$$\begin{aligned}\angle AH_2H_1 &= \angle AH_2O_1 + \angle O_1H_2H_1 \\ &= \angle O_1H_2H_1 + \angle H_1H_2O_3 = \angle O_1H_2O_3.\end{aligned}$$

It follows that another spiral similarity about H_2 takes triangle H_2AH_1 to triangle $H_2O_1O_3$. Hence $\angle O_3O_1H_2 = \angle H_1AH_2 = 90^\circ - \angle C$.

We wish to show that $\angle O_3T_2C = 90^\circ - \angle C$ as well, or in other words, $T_2O_3 \perp BC$. To do this, drop the altitude from O_3 to BC and let it intersect BC at D. Triangles ABC and H_1H_2C are similar as before, with corresponding incenters I and O_3. Furthermore, IT_2 and O_3D also correspond. Hence, $CT_2/T_2A = CD/DH_1$, and so $T_2D \parallel AH_1$. Thus, $T_2D \perp BC$, and it follows that $T_2O_3 \perp BC$.



Having shown that $O_1H_2T_2O_3$ is cyclic, we may now write $\angle O_1T_2O_3 = \angle O_1H_2O_3$. Since triangles H_2AO_1 and $H_2H_1O_3$ are related by a spiral similarity about H_2, we have

$$\angle O_1H_2O_3 = \angle AH_2H_1 = 180^\circ - \angle B,$$

by noting that ABH_2H_1 is cyclic. Likewise,

$$\angle O_2T_3O_1 = 180^\circ - \angle C \quad \text{and} \quad \angle O_3T_1O_2 = 180^\circ - \angle A,$$

and so $\angle O_1T_2O_3 + \angle O_2T_3O_1 + \angle O_3T_1O_2 = 360^\circ$. Therefore, $\angle T_3O_1T_2$, $\angle T_1O_2T_3$, and $\angle T_2O_3T_1$ of hexagon $O_1T_2O_3T_1O_2T_3$ also sum to 360°. Now let H be the intersection of circles ω_1 and ω_2. Then $\angle T_2HT_3 = 180^\circ - \frac{1}{2}\angle T_3O_1T_2$ and $\angle T_3HT_1 = 180^\circ - \frac{1}{2}\angle T_1O_2T_3$. Therefore,

$$\begin{aligned}\angle T_1HT_2 &= 360^\circ - \angle T_2HT_3 - \angle T_3HT_1\\ &= \frac{1}{2}\angle T_3O_1T_2 + \frac{1}{2}\angle T_1O_2T_3 = 180^\circ - \frac{1}{2}\angle T_1O_3T_2,\end{aligned}$$

and so H lies on the circle ω_3 as well. Hence, circles ω_1, ω_2, and ω_3 share a common point, as wanted.

Note. Readers might be nervous about the configurations, i.e., what if the hexagon $O_1T_2O_3T_1O_2T_3$ is not convex? Indeed, it is convex. It suffices to show that O_1, O_2, and O_3 are inside triangles AT_2T_3, BT_3T_1, and CT_1T_2, respectively. By symmetry, we only show that O_1 is inside AT_2T_3. Let d denote the distance from A to line T_2T_3. Then

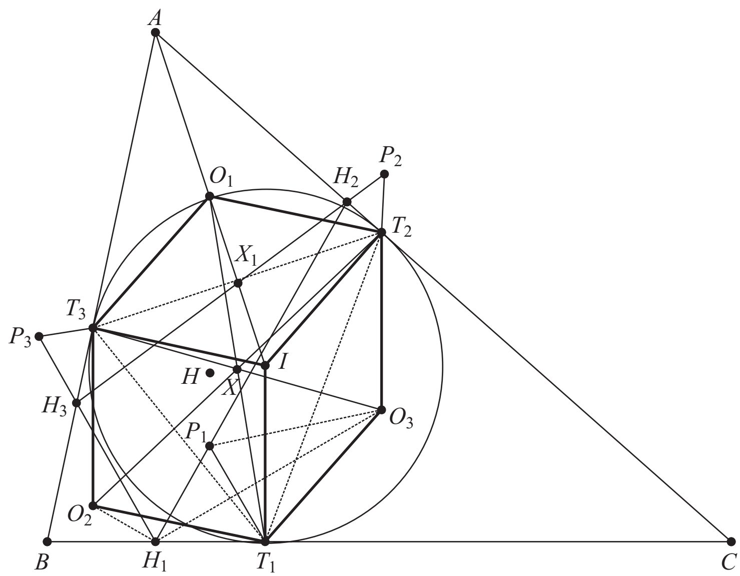
$$\frac{d}{AI} = \frac{d}{AT_2} \cdot \frac{AT_2}{AI} = \cos^2 \frac{\angle A}{2}.$$

On the other hand, triangles AH_2H_3 and ABC are similar with ratio $\cos \angle A$. Hence

$$\frac{AO_1}{AI} = \cos \angle A = 2\cos^2 \frac{\angle A}{2} - 1 \le \cos^2 \frac{\angle A}{2} = \frac{d}{AI},$$

by the **Double-angle formulas**. We conclude that O_1 is inside triangle AT_2T_3. Our second proof is based on above arguments.

Second Solution. (By Anders Kaseorg) Note that $AH_2 = AB\cos \angle A$ and $AH_3 = AC\cos \angle A$, so triangles AH_2H_3 and ABC are similar with ratio $\cos \angle A$. Thus, since O_1 is the incircle of triangle AH_2H_3, $AO_1 = AI\cos \angle A$. If X_1 is the intersection of segments AI and T_2T_3,



we have $\angle IX_1T_2 = \angle AT_2I = 90^\circ$, and so

$$X_1I = T_2I\cos\angle T_2IA = AI\cos^2\angle T_2IA = AI\sin^2\frac{\angle A}{2}$$
$$= AI\cdot\frac{1-\cos\angle A}{2} = \frac{AI - AO_1}{2} = \frac{O_1I}{2}.$$

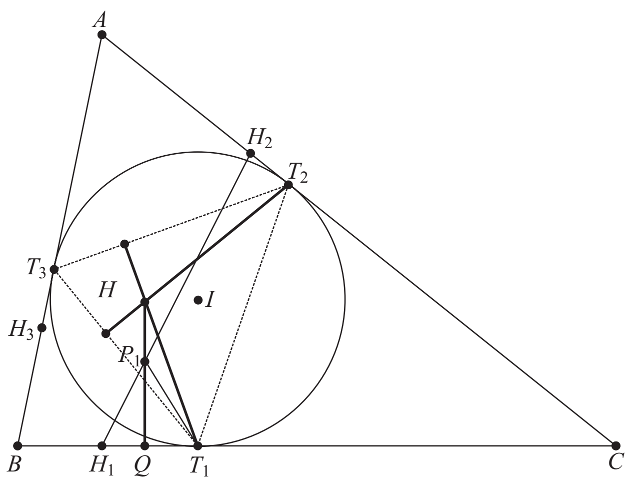
Hence $O_1X_1 = X_1I$, so O_1 is the reflection of I across line T_2T_3, and $O_1T_2 = IT_2 = IT_3 = O_1T_3$. Therefore, $O_1T_2IT_3$, and similarly $O_2T_3IT_1$ and $O_3T_1IT_2$, are rhombi with the same side length r, implying that circles $\omega_1, \omega_2, \omega$ have the same radius r. We also conclude that $O_1T_2 = T_3I = O_2T_1$ and $O_1T_2 \parallel T_3I \parallel O_2T_1$, and so $O_1O_2T_1T_2$ is a parallelogram. Hence the midpoints of O_1T_1 and O_2T_2 (similarly O_3T_3) are the same point P, and $O_1O_2O_3$ is the reflection of $T_1T_2T_3$ across P. If H is the reflection of I across P, we have $O_1H = O_2H = O_3H = r$, that is, H is a common point of the three circumcircles.

Note. Tony Zhang suggested the following finish. Because O_1 is the reflection of I across line T_2T_3 and I is the circumcenter of triangle $T_1T_2T_3$, $\angle T_3O_1T_2 = \angle T_2IT_3 = 2\angle T_2T_1T_3$. If H' is the orthocenter of triangle $T_1T_2T_3$, then

$$\angle T_2H'T_3 = 180^\circ - \angle T_2T_1T_3 = 180^\circ - \frac{\angle T_3O_1T_2}{2},$$

and so H' lies on ω_1. Similarly, H' lies on ω_2 and ω_3.

Third Solution. We use directed lengths (along line BC, with C to B as the positive direction) and directed angles modulo $180°$ in this proof. (For segments not lying on line BC, we assume its direction as the direction of its projection on line BC.) We claim that ω_i, $i = 1, 2, 3$, all pass through H, the orthocenter of triangle $T_1T_2T_3$. Without loss of generality, it suffices to prove that $T_1P_1T_2H$ is cyclic. If $AB = AC$, then $T_1 = H_1 = P_1$ and the case is trivial. Let $AB = c$, $BC = a$, $CA = b$, $\angle BAC = \alpha$, $\angle CBA = \beta$, and $\angle ACB = \gamma$.



Let Q be the intersection of lines HP_1 and BC. Note that

$$\begin{aligned}\angle HT_2T_1 &= 90° - \angle T_2T_1T_3\\ &= 90° - [180° - \angle T_3T_1B - \angle CT_1T_2]\\ &= 90° - \left[180° - \left(90° - \frac{\beta}{2}\right) - \left(90° - \frac{C}{2}\right)\right]\\ &= \frac{\alpha}{2}.\end{aligned}$$

(Likewise, $\angle T_2T_1H = \beta/2$.) Thus to prove that $T_1P_1T_2H$ is cyclic is equivalent to prove that $\angle QP_1T_1 = \alpha/2$.

Let Q_H and Q_P be the respective feet of perpendiculars from H and P_1 to line BC. Because $\angle AH_1B = \angle AH_2B = 90°$, ABH_1H_2 is cyclic, and so $\angle T_1H_1P_1 = \angle BH_1P_1 = \alpha$. Thus triangles AT_3T_2 and $H_1T_1P_1$

are similar, implying that

$$\angle Q_P P_1 T_1 = 90^\circ - \angle P_1 T_1 H_1 = 90^\circ - \left(90^\circ - \frac{\angle T_1 H_1 P_1}{2}\right) = \frac{\alpha}{2}.$$

Therefore, to prove that $\angle QP_1T_1 = \alpha/2$, we have now reduced to proving that $Q_P = Q_H$, or

$$\frac{T_1 Q_P}{T_1 H_1} = \frac{T_1 Q_H}{T_1 H_1}. \tag{1}$$

Note that

$$T_1 H_1 = P_1 H_1 \quad \text{and} \quad \frac{T_1 Q_P}{T_1 H_1} = 1 - \frac{Q_P H_1}{T_1 H_1},$$

that is,

$$\frac{T_1 Q_P}{T_1 H_1} = 1 - \frac{Q_P H_1}{P_1 H_1} = 1 - \cos \angle T_1 H_1 P_1 = 1 - \cos \alpha. \tag{2}$$

On the other hand, applying the **Law of Cosines** to triangle ABC gives

$$\begin{aligned} T_1 H_1 = T_1 C - H_1 C &= \frac{a+b-c}{2} - b\cos\gamma \\ &= \frac{a+b-c}{2} - \frac{a^2+b^2-c^2}{2a} = \frac{a(b-c)-(b^2-c^2)}{2a}, \end{aligned}$$

or

$$T_1 H_1 = \frac{(b-c)(a-b-c)}{2a} = \frac{(c-b)(b+c-a)}{2a}. \tag{3}$$

Now we calculate $T_1 Q_H$. Because H is the orthocenter of triangle $T_1T_2T_3$,

$$\begin{aligned} \angle T_1 H T_2 &= 180^\circ - \angle H T_2 T_1 - \angle T_2 T_1 H \\ &= (90^\circ - \angle H T_2 T_1) + (90^\circ - \angle T_2 T_1 H) \\ &= \angle T_2 T_1 T_3 + \angle T_3 T_2 T_1 = 180^\circ - \angle T_1 T_3 T_2. \end{aligned}$$

Applying the **Law of Sines** to triangle T_1T_2H and applying the **Extended Law of Sines** to triangle $T_1T_2T_3$ gives

$$\frac{T_1 H}{\sin \angle H T_2 T_1} = \frac{T_1 T_2}{\sin \angle T_1 H T_2} = \frac{T_1 T_2}{\sin \angle T_1 T_3 T_2} = 2r,$$

and consequently,

$$T_1 H = 2r \sin \angle H T_2 T_1 = 2r \sin \frac{\alpha}{2}.$$

Because

$$\angle Q_H T_1 H = \angle C T_1 T_2 + \angle T_2 T_1 H = \left(90^\circ - \frac{\gamma}{2}\right) + \frac{\beta}{2}$$
$$= 90^\circ + \frac{\beta - \gamma}{2},$$

we obtain

$$T_1 Q_H = T_1 H \cos \angle H T_1 Q_H = 2r \sin \frac{\alpha}{2} \sin \frac{\gamma - \beta}{2}. \tag{4}$$

Combining equations (1), (2), (3), and (4), we conclude that it suffices to prove that

$$1 - \cos\alpha = \frac{4ar \sin\frac{\alpha}{2} \sin\frac{\gamma-\beta}{2}}{(c-b)(b+c-a)}. \tag{5}$$

Applying the fact

$$\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} = \frac{r}{AT_2} = \frac{2r}{b+c-a},$$

and applying the Law of Sines to triangle ABC, (5) becomes

$$1 - \cos\alpha = \frac{2\sin\alpha \sin^2\frac{\alpha}{2} \sin\frac{\gamma-\beta}{2}}{\cos\frac{\alpha}{2}(\sin\gamma - \sin\beta)}. \tag{6}$$

By the **Double-angle formulas**, $1 - \cos\alpha = 2\sin^2\frac{\alpha}{2}$ and $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ and so (6) reads

$$\sin\gamma - \sin\beta = 2\sin\frac{\alpha}{2}\sin\frac{\gamma-\beta}{2}.$$

By the **Difference-to-product formulas**, the last equation reduces to

$$2\cos\frac{\beta+\gamma}{2}\sin\frac{\gamma-\beta}{2} = 2\sin\frac{\alpha}{2}\sin\frac{\gamma-\beta}{2},$$

which is evident.

3 IMO

1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\} \quad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

Note. The size $|S| = 10^6$ is unnecessarily large. See the second solution for a proof of the following stronger statement:

> If A is a k-element subset of $S = \{1, 2, \ldots, n\}$ and m is a positive integer such that $n > (m-1)\left(\binom{k}{2} + 1\right)$, then there exists $t_1, t_2, \ldots, t_m$ in S such that the sets $A_j = \{x + t_j \mid x \in A\}$, $j = 1, 2, \ldots, m$ are pairwise disjoint.

During the jury meeting, people decided to use the easier version as the first problem on the contest.

First Solution. Consider the set $D = \{x - y \mid x, y \in A\}$. There are at most $101 \times 100 + 1 = 10101$ elements in D (where the summand 1 represents the difference $x - y = 0$ for $x = y$). Two sets A_i and A_j have nonempty intersection if and only if $t_i - t_j$ is in D. It suffices to choose 100 numbers $t_1, t_2, \ldots, t_{100}$ in such a way that we do not obtain a difference from D.

We select these elements by induction. Choose one element arbitrarily. Assume that k elements, $k \leq 99$, have already been chosen. An element x that is already chosen prevents us from selecting any element from the set $x + D = \{x + d \mid d \in D\}$. Thus, after k elements are chosen, at most $10101k \leq 999999$ elements are forbidden. Hence we can select one more element. (Note that the numbers chosen are distinct because 0 is an element in D.)

Second Solution. (By Anders Kaseorg) We construct the set $\{t_j\}$ one element at a time using the following algorithm: Let $t_1 = 1 \in A$. For each j, $1 \leq j \leq 100$, let t_j be the smallest number in S that has not yet been crossed out, and then cross out t_j and all numbers of the form $t_j + |x - y|$ (with $x, y \in A$, $x \neq y$) that are in S. At each step, we cross out at most $1 + \binom{101}{2} = 5051$ new numbers. After picking t_1 through t_{99}, we have crossed out at most 500049 numbers, so there are always numbers in S that have not been crossed, so there are always candidates for t_j in S. (In fact, we will never need to pick a t_j bigger than 500050.)

Now, suppose A_j and A_k are not disjoint for some $1 \le j < k \le 100$. Then $x+t_j = y+t_k$ for some x, y in A. Since we cross out t_j immediately after picking it, $t_k \ne t_j$. Also, if $t_k < t_j$, we would have picked it on step j rather than step k (because $j < k$). Thus $t_k > t_j$, and so $x > y$. But this means that $t_k = t_j + x - y = t_j + |x-y|$, so t_k would have been crossed out on step j. This is a contradiction, so all sets A_j are pairwise disjoint.

2. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

Note. The answers are

$$(a, b) = (2t, 1) \quad \text{or} \quad (t, 2t) \quad \text{or} \quad (8t^4 - t, 2t)$$

for all positive integers t. It is routine to check the above are indeed solutions of the problem. We prove they are the only possible solutions. Assume that $a^2/(2ab^2 - b^3 + 1) = k$, where k is a positive integer. Then we have

$$a^2 = 2ab^2k - b^3k + k. \qquad (*)$$

We present three approaches.

First Solution. (Based on work by Anders Kaseorg) Rewrite equation $(*)$ as $a^2 - 2ab^2k = -b^3k + k$. Adding b^4k^2 to both sides completes the square on the left-hand side and gives

$$(kb^2 - a)^2 = b^4k^2 - b^3k + k,$$

or

$$(2kb^2 - 2a)^2 = (2b^2k)^2 - 2b(2b^2k) + 4k.$$

Completing the square on the right-hand side gives

$$(2kb^2 - 2a)^2 = (2b^2k - b)^2 + 4k - b^2.$$

or,

$$y^2 - x^2 = 4k - b^2,$$

where $x = 2kb^2 - b$ and $y = 2kb^2 - 2a$.

If $4k = b^2$, then either $x = y$ or $x = -y$. In the former case, $b = 2a$; in the latter case, $4kb^2 - b = 2a$, that is, $b^4 - b = 2a$. Because $k = b^2/4$

is an integer if and only if b is even, we get the solutions

$$\left(\frac{b}{2}, b\right) \quad \text{and} \quad \left(\frac{b^4 - b}{2}, b\right)$$

for any even b; that is, $(a, b) = (t, 2t)$ and $(a, b) = (8t^4 - t, 2t)$ for all positive integers t.

If $4k < b^2$, then $y^2 < x^2$, so $y^2 \le (x-1)^2$ (since x is clearly positive). Thus,

$$4k - b^2 \le (x-1)^2 - x^2 = -2x + 1 = -4kb^2 + 2b + 1,$$

or, $4k(b^2 + 1) \le b^2 + 2b + 1 < 3b^2 + 1$. Because $4k > 3$, this is a contradiction.

Similarly, if $4k > b^2$, then $y^2 > x^2$, so $y^2 \ge (x+1)^2$. Thus,

$$4k - b^2 \ge (x+1)^2 - x^2 = 2x + 1 = 4kb^2 - 2b + 1,$$

or $4k(b^2 - 1) + (b-1)^2 \le 0$. We must have $b = 1$ and

$$k = \frac{a^2}{2a - 1 + 1} = \frac{a}{2}.$$

This is an integer whenever a is even, so we get the solutions $(a, b) = (2t, 1)$ for all positive integers t.

Second Solution. (Based on work by Po-Ru Loh) Assume that $b = 1$. Then

$$\frac{a^2}{2ab^2 - b^3 + 1} = \frac{a^2}{2a} = \frac{a}{2}$$

is an positive integer if and only if a is even. Thus, $(a, b) = (2t, 1)$ are solutions of the problem for all positive integers t.

Now we assume that $b > 1$. Viewing equation $(*)$ as a quadratic in a, replace a by x to consider the equation

$$x^2 - 2b^2kx + (b^3 - 1)k = 0 \qquad (*')$$

for fixed positive integers b and k. Its roots are

$$x = \frac{2b^2k \pm \sqrt{4b^4k^2 - 4b^3k + 4k}}{2} = b^2k \pm \sqrt{b^4k^2 - b^3k + k}.$$

Assume that $x_1 = a$ is an integer root of equation $(*')$. Then $b^4k^2 - b^3k + k$ must be a perfect square. We claim that

$$\left(b^2k - \frac{b}{2} - \frac{1}{2}\right)^2 < b^4k^2 - b^3k + k < \left(b^2k - \frac{b}{2} + \frac{1}{2}\right)^2.$$

Note first that

$$\left(b^2k - \frac{b}{2} - \frac{1}{2}\right)^2 = b^4k^2 - 2b^2k\left(\frac{b}{2} + \frac{1}{2}\right)^2 + \frac{1}{4}(b+1)^2$$
$$= b^4k^2 - b^3k - b^2k + \frac{1}{4}(b+1)^2.$$

To establish the first inequality in our claim, it suffices to show that

$$-b^2k + \frac{1}{4}(b+1)^2 < k,$$

or, $(b+1)^2 < 4(b^2+1)k$, which is evident as $(b+1)^2 < 2(b^2+1)$ and $k \geq 1$.

Note also that

$$\left(b^2k - \frac{b}{2} + 1\right)^2 = b^4k^2 + 2b^2k\left(\frac{1}{2} - \frac{b}{2}\right) + \frac{1}{4}(1-b)^2$$
$$> b^4k^2 - b^3k + b^2k$$
$$> b^4k^2 - b^3k + k, \quad \text{as } b > 1,$$

which establishes the second inequality in our claim.

Because all of b, k, and $\sqrt{b^4k^2 - b^3k + k}$ are positive integers, we conclude from our claim that $b^2k - \frac{b}{2}$ is an integer and that

$$b^4k^2 - b^3k + k = \left(b^2k - \frac{b}{2}\right)^2 = b^4k - b^3k + \frac{b^2}{4},$$

and so $k = b^2/4$. Thus, $b = 2t$ for some positive integer t. The two solution of the equation $(*')$ becomes

$$x = b^2k \pm \left(b^2k - \frac{b}{2}\right),$$

that is. $x = t$ or $x = 8t^4 - t$. Hence, $(a,b) = (t,2t)$ and $(a,b) = (8t^4 - t, 2t)$ are the possible solutions of the problem, in addition to the solutions $(2t,1)$.

Third Solution. Because both k and a^2 are positive, $2ab^2 - b^3 + 1 > 0$, or,

$$2a > b - \frac{1}{b^2}.$$

Because a and b are positive integers, we have $2a \geq b$. Because k is a positive integer, $a^2 \geq 2ab^2 - b^3 + 1$, or, $a^2 \geq b^2(2a-b) + 1$. Because

$2a-b \geq 0$ and $a^2 > b^2(2a-b) \geq 0$, we have

$$a > b \quad \text{or} \quad 2a = b. \tag{$\dagger$}$$

We consider again the quadratic equation $(*')$ for fixed positive integers b and k, and assume that $x_1 = a$ is an integer root of equation $(*')$. Then the other root x_2 is also an integer because $x_1 + x_2 = 2b^2k$. Without loss of generality, we assume that $x_1 \geq x_2$. Then $x_1 \geq b^2k > 0$. Furthermore, because $x_1x_2 = (b^3-1)k$, we obtain

$$0 \leq x_2 = \frac{(b^3-1)k}{x_1} \leq \frac{(b^3-1)k}{b^2k} < b.$$

If $x_2 = 0$, then $b^3 - 1 = 0$, and so $x_1 = 2k$ and (a, b) can be written in the form of $(2t, 1)$ for some integers t.

If $x_2 > 0$, then $(a, b) = (x_2, b)$ is a pair of positive integers satisfying the equations $(\dagger)$ and $(*)$. We conclude that $2x_2 = b$, and so

$$k = \frac{x_2^2}{2x_2b^2 - b^3 + 1} = x_2^2 = \frac{b^2}{4},$$

and $x_1 = b^4/2 - b/2$. Thus, (a, b) can be written in the form of either $(t, 2t)$ or $(8t^3 - t, 2t)$ for some positive integers t.

3. A convex hexagon is given in which any two opposite sides have the following property: the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

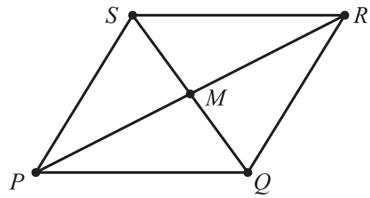
(A convex $ABCDEF$ has three pairs of opposite sides: AB and DE, BC and EF, CD and FA.)

Note. We present three solutions. The first two apply vector calculations, while the last two are more synthetic. The solutions investigate the angles formed by the three main diagonals. All solutions are based on the following closely related geometric facts.

Lemma 1a. *Let $PQRS$ be a parallelogram. If $PR \geq \sqrt{3}QS$, then $\angle SPQ \leq 60^\circ$ with equality if and only $PQRS$ is a rhombus.*

Proof. Let $PQ = x$, $QR = PS = y$, $\angle SPQ = \alpha$. Then $\angle PQR = 180^\circ - \alpha$. Applying the **Law of Cosines** to triangles PQR and PQS gives

$$PR^2 = x^2 + y^2 - 2xy\cos\angle PQR = x^2 + y^2 + 2xy\cos\alpha$$



and $QS^2 = x^2 + y^2 - 2xy\cos\alpha$. The condition $PR \geq \sqrt{3}QS$ becomes

$$x^2 + y^2 + 2xy\cos\alpha \geq 3(x^2 + y^2 - 2xy\cos\alpha),$$

or,

$$4xy\cos\alpha \geq x^2 + y^2.$$

Because $x^2 + y^2 \geq 2xy$, we conclude that $\cos\alpha \geq \frac{1}{2}$, that is, $\alpha \leq 60°$. Equality holds if and only if $x = y$, that is, $PQRS$ is a rhombus. ■

If we only look at half of the parallelogram—triangle PQS—then Lemma 1a leads to the following.

Lemma 1b. *In triangle PQS, let M be the midpoint of side QS. If $2PM \geq \sqrt{3}QS$, then $\angle SPQ \leq 60°$. Equality holds if and only PQS is equilateral.*

We can also rewrite Lemma 1a in the language of vectors as the following.

Lemma 1c. *Let $\mathbf{v}$ and $\mathbf{u}$ be two vectors in the plane. If*

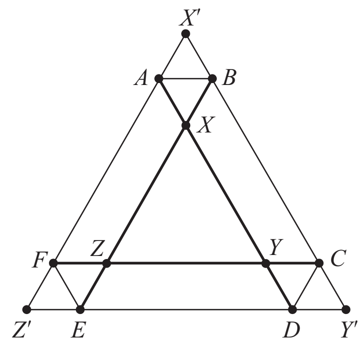
$$|\mathbf{u} + \mathbf{v}| \geq \sqrt{3}(|\mathbf{u} - \mathbf{v}|),$$

then the angle formed by $\mathbf{u}$ and $\mathbf{v}$ is no greater than $60°$, and equality holds if and only if $|\mathbf{u}| = |\mathbf{v}|$.

Let X, Y, Z be the intersections of the diagonals of the quadrilateral $ABCDEF$, as shown. All of the solutions use the following lemma.

Lemma 2. *Let $ABCDEF$ be a convex hexagon with parallel opposite sides, that is, $AB \parallel DE$, $BC \parallel EF$, and $CD \parallel FA$. Assume that each pair of three diagonals AD, BE, CF form a $60°$ angle and that $AD = BE = CF$. Then the hexagon is equal angular. Furthermore, the hexagon can be obtained by cutting three congruent triangles from each corner of a equilateral triangle.*

Proof. Because $AB \parallel DE$, triangles XAB and XDE are similar. This implies that $XA - XB$ and $XD - XE$ have the same sign. But since $AD = EB$, we also have $XA - XB = -(XD - XE)$. Thus $XA = XB$



and $XD = XE$. Because $\angle AXB = 60°$, triangle XAB is equilateral, so $\angle ABE = 60°$. In the same way we can show that $\angle EBC = 60°$. Thus $\angle ABC = 120°$. Similarly all the other angles of the hexagon measure 120°.

Let X', Y', Z' be the intersections of lines BC, DE, and FA. It is not difficult to see that triangles ADZ', $\angle BEY'$, and CFX' are congruent equilateral triangles, and consequently, the hexagon is obtained by cutting congruent equilateral triangles $X'AB$, $Y'CD$, and $Z'EF$ from equilateral triangle $X'Y'Z'$. This completes our proof. ■

First Solution. (Based on work by Anders Kaseorg) Choose an arbitrary point O as the origin. Let each lowercase letter denote the vector from O to the point labeled with the corresponding uppercase letter. We are given:

$$\left|\frac{a+b}{2} - \frac{d+e}{2}\right| = \frac{\sqrt{3}}{2}\left(|b-a| + |d-e|\right).$$

Thus, by the **Triangle Inequality**, we have

$$\begin{aligned}|(b-e)+(a-d)| &= |a+b-d-e| \\ &= \sqrt{3}\left(|b-a|+|d-e|\right) \\ &\geq \sqrt{3}|b-a+d-e| \\ &= \sqrt{3}|(b-e)-(a-d)|,\end{aligned}$$

and inequality holds if and only if vectors $b-a$ and $d-e$ are differ by a positive scale multiple. In other words,

$$|\overrightarrow{EB} + \overrightarrow{DA}| \geq \sqrt{3}|\overrightarrow{EB} - \overrightarrow{DA}|$$

and equality holds if and only if $\overrightarrow{AB} = k\overrightarrow{ED}$ for some real number $k > 0$. By Lemma 1c, we conclude that $\angle ZXY = \angle AXB \le 60^\circ$.

Analogously, we can show that $\angle XYZ, \angle YZX \le 60^\circ$. But this can only happen if all of these angles measure 60°. Hence all the equalities hold in the above discussions. Therefore, by Lemma 1c, each pair of diagonals AD, BE, and CF form a 60° angle and $AD = CF = EB$. Because $\overrightarrow{AB} = k\overrightarrow{ED}$ for some real number $k > 0$, $AB \parallel ED$, Likewise, $BC \parallel DF$ and $CD \parallel FA$. The desired result now follows from Lemma 2.

Note. The step rearranging $|a + b - d - e|$ as $|(b - e) + (a - d)|$ seems rather tricky. The following approach reveals the importance of vectors $a - d$, $b - e$, and $c - f$. Again by the Triangular Inequality, we have

$$|a + b - d - e| \ge \sqrt{3}|a - b + e - d|,$$

or,

$$(a + b - d - e) \cdot (a + b - d - e) \ge 3(a - b + e - d) \cdot (a - b + e - d),$$

where $\cdot$ represents the **Dot Product** of vectors. Expanding the above equation and collecting the likely terms gives

$$a \cdot a + b \cdot b + d \cdot d + e \cdot e - 4a \cdot b - 4 \cdot e + 4a \cdot e + 4b \cdot d - 2a \cdot d - 2b \cdot e \le 0.$$

Adding the analogous results from the given conditions on other pairs of opposite sides yields

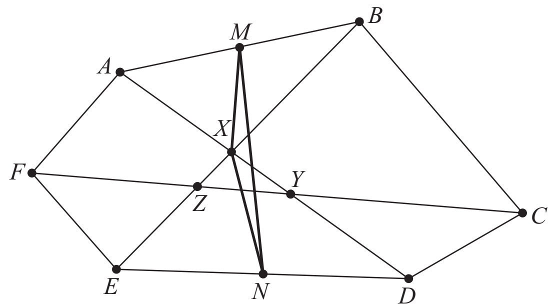
$$\sum_{\text{cyc}} (a \cdot a - 2a \cdot b + 2a \cdot c - a \cdot d) \le 0,$$

where $\sum_{\text{cyc}}$ is the **cyclic sum** of variables (a, b, c, d, e, f). Note that the left-hand side of the above inequality is

$$(a - b + c - d + e - f) \cdot (a - b + c - d + e - f) = |a - b + c - d + e - f|^2.$$

Thus, all equalities hold for all the above inequalities. In particular, $AB \parallel DE$ and $a - b = c - d + e - f = 0$, or, $c - f = -(a - b + e - d)$. We conclude that $AB \parallel DE \parallel CF$ and that $|a + b - d - e| = \sqrt{3}CF$. Because the given conditions are cyclic, it is now natural to consider the other two diagonals AD and BE, by rewriting $a + b - d - e = (b - e) + (a - d) = \overrightarrow{EB} + \overrightarrow{DA}$.

Second Solution. (Based on work by Po-Ru Loh) Angles AXB, CYD, EZF are the vertical angles of triangle XYZ (which may be degenerated to a point), so the largest of these three angles is at least 60°. Without loss of generality, assume that $\angle AXB$ is the largest angle, and so $\angle AXB =$



$\angle DXE \geq 60^\circ$. By Lemma 1b, we have $2XM \leq \sqrt{3}AB$ and $2XN \leq \sqrt{3}DE$, implying that

$$XM + XN \leq \frac{\sqrt{3}}{2}(AB + DE).$$

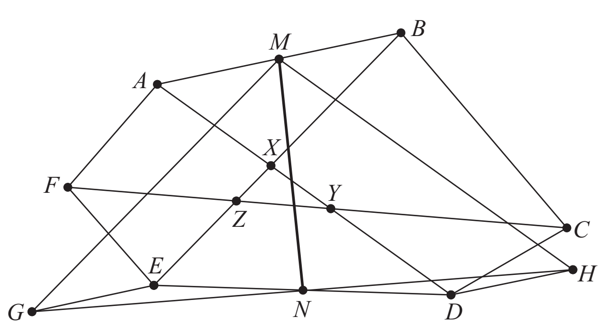
By the Triangle Inequality, we have

$$MN \leq XM + XN \leq \frac{\sqrt{3}}{2}(AB + DE).$$

By the given condition, all equalities hold in our discussions above. Thus, all the conditions of Lemma 2 are satisfied, from which our desired result follows.

Third Solution. (Based on the comments from Svetoslav "Beto" Savchev, member of the IMO Problem Selection Committee) We want to "add" the length of AB and DE without violating the midpoints constrain.

Let G and H be points such that $AMHD$ and $BMGE$ are parallelograms. Thus, $AD = MH$, $BE = MG$, $AD \parallel MH$, and $BE \parallel MG$. We conclude that $\angle AXB = \angle GMH$ and that $MG = MH$ if and only if $AD = BE$.



Because N is the midpoint of segment DE, it is not hard to see that $GEHD$ is also a parallelogram, and so N is the midpoint of segment GH. Note that $AB + ED = GE + ED + DH \geq GH$, and equality holds if and only if $AB \parallel DE$. In triangle MGH, median MN is at least $\sqrt{3}/2$ opposite side GH. By Lemma 1b, we conclude that $\angle GMH \geq 60^\circ$, that is, $\angle AXB = \angle ZXY \geq 60^\circ$. Likewise, $\angle XYZ, \angle YZX \geq 60^\circ$. Thus, all inequalities hold, implying that all the conditions of Lemma 2 hold, from which our desired follows.

4. Let $ABCD$ be a convex quadrilateral. Let P, Q and R be the feet of perpendiculars from D to lines BC, CA and AB, respectively. Show that $PQ = QR$ if and only if the bisectors of angles ABC and ADC meet on segment AC.

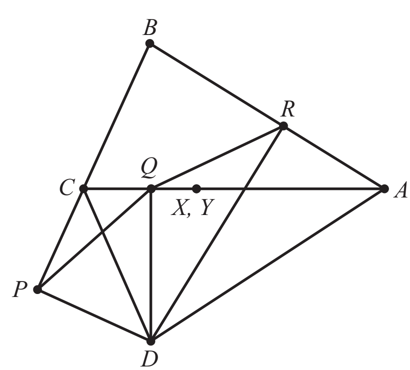
Note. The condition that $ABCD$ be cyclic is not necessary.

Solution. As usual, we set $\angle ABC = \beta$, $\angle BCA = \gamma$, and $\angle CAB = \alpha$. Because $\angle DPC = \angle CQD = 90^\circ$, quadrilateral $CPDQ$ is cyclic with CD a diameter of the circumcircle. By the **Extended Law of Sines**, we have $PQ = CD \sin \angle PCQ = CD \sin(180^\circ - \gamma) = CD \sin \gamma$. Likewise, by working with cyclic quadrilateral $ARQD$, we find $RQ = AD \sin \alpha$. Hence, $PQ = RQ$ if and only if $CD \sin \gamma = AD \sin \alpha$. Applying the Law of Sines to triangles BAC, we conclude that

$$PQ = RQ \quad \text{if and only if} \quad \frac{AB}{BC} = \frac{AD}{CD}. \qquad (*)$$

On the other hand, let bisectors of $\angle CBA$ and $\angle ADC$ meet segment AC at X and Y, respectively. By the **Angle-bisector Theorem**, we have

$$\frac{AX}{CX} = \frac{AB}{BC} \quad \text{and} \quad \frac{AY}{CY} = \frac{AD}{CD}.$$



Hence, the bisectors of $\angle ABC$ and $\angle ADC$ meet on segment AC if and only if $X = Y$, or,

$$\frac{AB}{BC} = \frac{AX}{CX} = \frac{AY}{CY} = \frac{AD}{CD}. \qquad (**)$$

Our desired result follows from relations $(*)$ and $(**)$.

5. Let n be a positive integer and $x_1, x_2, \dots, x_n$ be real numbers with $x_1 \le x_2 \le \cdots \le x_n$.
   (a) Prove that
   $$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\right)^2 \le \frac{2(n^2-1)}{3}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$
   (b) Show that the equality holds if and only if $x_1, x_2, \dots, x_n$ form an arithmetic sequence.

Note. The desired inequality fits well in the format of the **Cauchy–Schwarz Inequality**. Part (b) also indicates that the equality case of Cauchy-Schwarz Inequality holds if and only if $x_i - x_j = d(i-j)$, that is, $\frac{x_i - x_j}{i-j} = d$. Thus, it is natural to explore the the relation between

$$\frac{2(n^2-1)}{3} \quad \text{and} \quad \sum_{i,j=1}^{n}(i-j)^2.$$

Because part (b) helps to solve this problem, people suggested removing this part during the jury meeting at the IMO. After some careful discussions, people decided to leave it as it is. In authors' view, it might be better (and certainly more difficult) to ask the question in the following way:

> Let n be a positive integer and $x_1, x_2, \dots, x_n$ be real numbers with $x_1 \le x_2 \le \cdots \le x_n$. Determine the smallest constant c, in terms of n, such that
> $$\left(\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|\right)^2 \le c\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2.$$

Solution. We adapt double sum notation

$$\sum_{i,j=1}^{n} \quad \text{for} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}.$$

The desired inequality reads

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

By Cauchy–Schwarz Inequality, we have

$$\left(\sum_{i,j=1}^{n} (x_i - x_j)^2\right)\left(\sum_{i,j=1}^{n} (i-j)^2\right) \geq \left(\sum_{i,j=1}^{n} |i-j||x_i - x_j|\right)^2.$$

It suffices to show that

$$\sum_{i,j=1}^{n} (i-j)^2 = \frac{n^2(n^2-1)}{6} \tag{†}$$

and

$$\left(\sum_{i,j=1}^{n} |i-j||x_i - x_j|\right)^2 = \frac{n^2}{4}\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2,$$

or

$$\sum_{i,j=1}^{n} |i-j||x_i - x_j| = \frac{n}{2}\sum_{i,j=1}^{n} |x_i - x_j|. \tag{‡}$$

Note that

$$\begin{aligned}
\sum_{i,j=1}^{n} (i-j)^2 &= \sum_{i,j=1}^{n} (i^2 + j^2 - 2ij) = \sum_{i,j=1}^{n} (i^2 + j^2) - 2\sum_{i=1}^{n}\sum_{j=1}^{n} ij \\
&= 2\sum_{i,j=1}^{n} i^2 - 2\left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j\right) \\
&= 2\sum_{i=1}^{n}\sum_{j=1}^{n} i^2 - 2\left(\frac{n(n+1)}{2}\right)^2 = 2\sum_{i=1}^{n} ni^2 - \frac{n^2(n+1)^2}{2} \\
&= 2n \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{2} \\
&= n^2(n+1) \cdot \frac{2(2n+1) - 3(n+1)}{6} \\
&= n^2(n+1)\frac{n-1}{6} = \frac{n^2(n^2-1)}{6},
\end{aligned}$$

establishing identity (†).

To establish identity (‡), we compare the coefficients of x_i, $1 \le i \le n$, on both sides of the identity. The coefficient of x_i on the left-hand side is equal to

$$\begin{aligned}(i-1)+(i-2)+\cdots+[(i-(i-1)]-[(i+1)-i]-\cdots-(n-i)\\ &= \frac{i(i-1)}{2}-\frac{(n-i)(n-i+1)}{2}\\ &= \frac{i(i-1)-n^2+(2i-1)n-i(i-1)}{2}\\ &= \frac{n(2i-n-1)}{2}.\end{aligned}$$

On the other hand, the coefficient of x_i on the right-and side is equal to

$$\frac{n}{2}[(\underbrace{1+1+\cdots+1}_{i-1 \text{ times}})-(\underbrace{1+1+\cdots+1}_{n-i \text{ times}})]=\frac{n}{2}(2i-n-1).$$

Therefore, identity (‡) is true and the proof of part (a) is complete.

There is only one inequality step (when we applied Cauchy-Schwarz Inequality) in our proof of the desired inequality. The equality holds if and only if the Cauchy-Schwarz Inequality reaches equality, that is,

$$\frac{x_i-x_j}{i-j}=d$$

is a constant for $1 \le i, j \le n$. In particular, $x_i - x_1 = d(i-1)$, that is, $x_1, x_2, \ldots, x_n$ is an arithmetic sequence.

6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

Solution. We approach indirectly by assuming that such q does not exist. Then for any fixed prime q, there is a positive integer n such that $n^p - p$ is divisible by q, that is

$$n^p \equiv p \pmod q. \tag{$*$}$$

If q divides n, then q divides p, and so $q = p$. We further assume that $q \ne p$. Hence q does not divide n. We start with the following well-known fact.

Lemma. *Let q be a prime, and let n be a positive integer relatively prime to q. Denote by d_n the order of n modulo q, that is, d_n is the smallest* *positive integer such that* $n^{d_n} \equiv 1 \pmod{q}$. *Then for any positive integer* m *such that* $n^m \equiv 1 \pmod{q}$, d_n *divides* m.

Proof. By the minimality of d_n, we can write $m = d_n k + r$ where k and r are integers with $1 \le k$ and $0 \le r < d_n$. Then

$$1 \equiv n^m \equiv n^{d_n k + r} \equiv n^{d_n k} \cdot n^r = n^r \pmod{q}.$$

By the minimality of d_n, $r = 0$, that is, d_n divides m. ■

By **Fermat's little Theorem**, $n^{q-1} \equiv 1 \pmod{q}$. Thus, by the Lemma, d_n divides $q - 1$. For the positive integer n, because $n^p \equiv p \pmod{q}$, we have $n^{pd_p} \equiv p^{d_p} \equiv 1 \pmod{q}$. Thus, by the Lemma, d_n divides both $q - 1$ and pd_p, implying that d_n divides $\gcd(q - 1, pd_p)$.

Now we pick a prime q such that (a) q divides $\frac{p^p-1}{p-1} = 1+p+\cdots+p^{p-1}$, and (b) p^2 does not divide $q - 1$. First we show that such a q does exist. Note that $1 + p + \cdots + p^{p-1} \equiv 1 + p \not\equiv 1 \pmod{p^2}$. Hence there is a prime divisor of $1 + p + \cdots + p^{p-1}$ that is not congruent to 1 modulo p^2, and we can choose that prime to be our q.

By (a), $p^p \equiv 1 \pmod{q}$ (and $p \ne q$). By the Lemma, d_p divides p, and so $d_p = p$ or $d_p = 1$.

If $d_p = 1$, then $p \equiv 1 \pmod{q}$.

If $d_p = p$, then d_n divides $\gcd(p^2, q - 1)$. By (b), the possible values of d_n are 1 and p, implying that $n^p \equiv 1 \pmod{q}$. By relation $(*)$, we conclude $p \equiv 1 \pmod{q}$.

Thus, in any case, we have $p \equiv 1 \pmod{q}$. But then by (a), $0 \equiv 1 + p + \cdots + p^{p-1} \equiv p \pmod{q}$, implying that $p = q$, which is a contradiction. Therefore our original assumption was wrong, and there is a q such that for every integer n, the number $n^p - p$ is not divisible by q.

Note. The proof can be shortened by starting directly with the definition of q as in the second half of the above proof. But we think the argument in the first part provides motivation for the choice of this particular q.

Many students were able to apply Fermat's Little Theorem to realize that $n^{pd_p} \equiv p^{d_p} \equiv 1 \pmod{q}$. It is also not difficult to see that there are integers n such that $n^{d_p} \not\equiv 1 \pmod{q}$, because of the existence of primitive roots modulo q. By the minimality of d_p, we conclude that $d_p = pk$, where k is some divisor of d_p. Consequently, we have $pk \mid (q-1)$, implying that $q \equiv 1 \pmod{p}$. This led people to think about various applications of **Dirichlet's Theorem**, which is an very popular but fatal approach to this problem. However, a solution with advanced mathematics background is available. It involves a powerful prime density theorem. The prime q satisfies the required condition if and only if q remains a prime in the field $k = \mathbb{Q}(\sqrt[p]{p})$. By applying Chebotarev's density theorem to the Galois closure of k, we can show that the set of such q has density $\frac{1}{p}$, implying that there are infinitely many q satisfying the required condition. Of course, this approach is far beyond the knowledge of most IMO participants.

4
Problem Credits

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2. Gregory Galperin
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4. Titu Andreescu and Zuming Feng
5. Titu Andreescu and Zuming Feng
6. Reid Barton

Team Selection Test

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2. Titu Andreescu
3. Reid Barton
4. Alex Saltman
5. Titu Andreescu
6. Zuming Feng

IMO

1. Brazil
2. Bulgaria
3. Poland
4. Finland
5. Ireland
6. France

5
Glossary

Angle Bisector Theorem Let ABC be a triangle, and let D be a point of side BC such that segment AD bisects $\angle BAC$. Then

$$\frac{AB}{AC} = \frac{BD}{DC}.$$

Cauchy–Schwarz Inequality For any real numbers $a_1, a_2, \ldots, a_n$, and $b_1, b_2, \ldots, b_n$

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \\ \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2,$$

with equality if and only if a_i and b_i are proportional, $i = 1, 2, \ldots, n$.

Ceva's theorem and its trigonometric form Let AD, BE, CF be three **cevians** of triangle ABC. The following are equivalent:

(i) AD, BE, CF are concurrent;

(ii) $\dfrac{AF}{FB} \cdot \dfrac{BD}{DC} \cdot \dfrac{CE}{EA} = 1;$

(iii) $\dfrac{\sin\angle ABE}{\sin\angle EBC} \cdot \dfrac{\sin\angle BCF}{\sin\angle FCA} \cdot \dfrac{\sin\angle CAD}{\sin\angle DAB} = 1.$

Cevian A cevian of a triangle is any segment joining a vertex to a point on the opposite side.

Cyclic Sum Let n be a positive integer. Given a function f of n variables, define the cyclic sum of variables $(x_1, x_2, \ldots, x_n)$ as

$$\sum_{\text{cyc}} f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) + f(x_2, x_3, \ldots, x_n, x_1) + \cdots + f(x_n, x_1, x_2, \ldots, x_{n-1}).$$

Dirichlet's Theorem A set S of primes is said to have *Dirichlet density* if

$$\lim_{s \to 1} \frac{\sum_{p \in S} p^{-s}}{\ln(s-1)^{-1}}$$

exists, where $\ln$ denotes the natural logarithm. If the limit exists we denote it by $d(S)$ and call $d(S)$ the Dirichlet density of S.

There are infinitely many primes in any arithmetic sequence of integers for which the common difference is relatively prime to the terms. In other words, let a and m be relatively prime positive integers, then there are infinitely many primes p such that $p \equiv a \pmod{m}$. More precisely, let $S(a; m)$ denote the set of all such primes. Then $d(S(a; m)) = 1/\phi(m)$, where ϕ is Euler's function.

Dot Product Let n be an integer greater then 1, and let $\mathbf{u} = [a_1, a_2, \ldots, a_n]$ and $\mathbf{v} = [b_1, b_2, \ldots, b_n]$ be two vectors. Define their *dot product* $\mathbf{u} \cdot \mathbf{v} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$. It is easy to check that

(i) $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$, that is, the dot product of vector with itself is the square of the magnitude of $\mathbf{v}$ and $\mathbf{v} \cdot \mathbf{v} \ge 0$ with equality if and only if $\mathbf{v} = [0, 0, \ldots, 0]$;

(ii) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$;

(iii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, where $\mathbf{w}$ is a vector;

(iv) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$, where c is a scalar.

When vectors $\mathbf{u}$ and $\mathbf{v}$ are placed tail-by-tail at the origin O, let A and B be the tips of $\mathbf{u}$ and $\mathbf{v}$, respectively. Then $\overrightarrow{AB} = \mathbf{v} - \mathbf{u}$. Let $\angle AOB = \theta$. Applying the **Law of Cosines** to triangle AOB yields

$$\begin{aligned} |\mathbf{v} - \mathbf{u}|^2 &= AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cos\theta \\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}| \cos\theta. \end{aligned}$$

It follows that

$$(\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2|\mathbf{u}||\mathbf{v}| \cos\theta,$$

or,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Consequently, if $0 \le \theta \le 90^\circ$, $\mathbf{u}\cdot\mathbf{v} \ge 0$. Considering the range of $\cos\theta$, we have provided a proof of the **Cauchy–Schwarz Inequality**.

Extended Law of Sines In a triangle ABC with circumradius equal to R,

$$\frac{BC}{\sin A} = \frac{CA}{\sin B} = \frac{AB}{\sin C} = 2R.$$

Fermat's Little Theorem If p is prime, then $a^p \equiv a \pmod{p}$ for all integers a.

Law of Cosines In a triangle ABC,

$$CA^2 = AB^2 + BC^2 - 2AB\cdot BC\cos\angle ABC,$$

and analogous equations hold for AB^2 and BC^2.

Pigeonhole Principle If n objects are distributed among $k < n$ boxes, some box contains at least two objects.

Primitive Root Let n be a positive integer. An integer a is called a primitive root modulo n if a and n are relatively prime and $\phi(n)$ is the smallest positive integer such that $a^{\phi(n)} \equiv 1 \pmod{n}$. Integer n possesses primitive roots if and only if n is of the form $2, 4, p^m, 2p^m$, where p is an odd prime and m is a positive integer. If $n = p$ is a prime, then $\phi(p) = p - 1$. Equivalently, an integer a is a primitive root modulo p if and only if $a, a^2, \ldots, a^{p-1}$ are all distinct modulo p, that is,

$$\{a, a^2, \ldots, a^{p-1}\} \equiv \{1, 2, \ldots p-1\} \pmod{p}.$$

Schur's Inequality Let x, y, z be nonnegative real numbers. Then for any $r > 0$,

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \ge 0.$$

Equality holds if and only if $x = y = z$ or if two of x, y, z are equal and the third is equal to 0.

The proof of the inequality is rather simple. Because the inequality is symmetric in the three variables, we may assume without loss of generality that $x \ge y \ge z$. Then the given inequality may be rewritten as

$$(x-y)[x^r(x-z) - y^r(y-z)] + z^r(x-z)(y-z) \ge 0,$$

and every term on the left-hand side is clearly nonnegative. The first term is positive if $x > y$, so equality requires $x = y$, as well as $z^r(x-z)(y-z) = 0$, which gives either $x = y = z$ or $z = 0$.

Spiral Similarity See **transformation**.

Transformation A transformation of the plane is a mapping of the plane onto itself such that every point P is mapped into a unique image P' and every point Q' has a unique prototype (preimage, inverse image, counterimage) Q.

A **reflection across a line** (in the plane) is a transformation which takes every point in the plane into its mirror image, with the line as mirror. A **rotation** is a transformation when the entire plane is rotated about a fixed point in the plane.

A **similarity** is a transformation that preserves ratios of distances. If P' and Q' are the respective images of points P and Q under a similarity $\mathbf{T}$, then the ratio $P'Q'/PQ$ depends only on $\mathbf{T}$. This ratio is the **similitude** of $\mathbf{T}$. A **dilation** is a direction-preserving similarity, i.e., a similarity that takes each line into a parallel line.

The **product** $\mathbf{T}_2 T_1$ **of two transformations** is transformation defined by $\mathbf{T}_2 T_1 = T_2 \circ T_1$, where $\circ$ denotes function composition. A **spiral similarity** is the product of a rotation and a dilation, or vice versa.

Triangle Inequality Let $z = a + bi$ be a complex number. Define the absolute value of z to be

$$|z| = \sqrt{a^2 + b^2}.$$

Let α and β be two complex numbers. The inequality

$$|\alpha + \beta| \le |\alpha| + |\beta|$$

is called the triangle inequality.

Let $\alpha = \alpha_1 + \alpha_2 i$ and $\beta = \beta_1 + \beta_2 i$, where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers. Then $\alpha + \beta = (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2)i$. Vectors $\mathbf{u} = [\alpha_1, \alpha_2]$, $\mathbf{v} = [\beta_1, \beta_2]$, and $\mathbf{w} = [\alpha_1 + \beta_1, \alpha_2 + \beta_2]$ form a triangle with sides lengths $|\alpha|$, $|\beta|$, and $|\alpha + \beta|$. The triangle inequality restates the fact the the length of any side of a triangle is less than the sum of the lengths of the other two sides.

Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1,$$

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x}, & \cot x &= \frac{1}{\tan x}, \\
\sin(-x) &= -\sin x, & \cos(-x) &= \cos x, \\
\tan(-x) &= -\tan x, & \cot(-x) &= -\cot x, \\
\sin(90^\circ \pm x) &= \cos x, & \cos(90^\circ \pm x) &= \mp \sin x, \\
\tan(90^\circ \pm x) &= \mp \cot x, & \cot(90^\circ \pm x) &= \mp \tan x, \\
\sin(180^\circ \pm x) &= \mp \sin x, & \cos(180^\circ \pm x) &= -\cos x, \\
\tan(180^\circ \pm x) &= \pm \tan x, & \cot(180^\circ \pm x) &= \pm \cot x.
\end{aligned}$$

Addition and subtraction formulas:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}.$$

Double-angle formulas:

$$\sin 2a = 2 \sin a \cos a,$$

$$\cos 2a = 2\cos^2 a - 1 = 1 - 2\sin^2 a = \cos^2 \alpha - \sin^2 \alpha,$$

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}.$$

Triple-angle formulas:

$$\sin 3a = 3 \sin a - 4 \sin^3 a = (3 - 4\sin^2 a) \sin a = (4 \cos^2 a - 1) \sin a,$$

$$\cos 3a = 4 \cos^3 a - 3 \cos a = (4 \cos^2 -3) \cos a = (1 - 4 \sin^2 a) \cos a,$$

$$\tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a}.$$

Half-angle formulas:

$$\sin^2 \frac{a}{2} = \frac{1 - \cos a}{2},$$

$$\cos^2 \frac{a}{2} = \frac{1 + \cos a}{2}.$$

Sum-to-product formulas:

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2},$$

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2},$$

$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a\cos b}.$$

Difference-to-product formulas:

$$\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2},$$

$$\cos a - \cos b = -2\sin\frac{a-b}{2}\sin\frac{a+b}{2},$$

$$\tan a - \tan b = \frac{\sin(a-b)}{\cos a\cos b}.$$

Product-to-sum formulas:

$$2\sin a\cos b = \sin(a+b) + \sin(a-b),$$

$$2\cos a\cos b = \cos(a+b) + \cos(a-b),$$

$$2\sin a\sin b = -\cos(a+b) + \cos(a-b).$$

Weighted AM-GM Inequality If $a_1, a_2, \dots, a_n$ are n nonnegative real numbers, and if m_1, m_2, $\dots$, m_n are positive real numbers satisfying

$$m_1 + m_2 + \cdots + m_n = 1,$$

then

$$m_1a_1 + m_2a_2 + \cdots + m_na_n \ge a_1^{m_1}a_2^{m_2}\cdots a_n^{m_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

6

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7
Appendix

1 2003 Olympiad Results

Tiankai Liu and Po-Ru Loh, both with perfect scores, were the winners of the Samuel Greitzer-Murray Klamkin award, given to the top scorer(s) on the USAMO. Mark Lipson placed third on the USAMO. They were awarded college scholarships of $5000, $5000, $2000, respectively, by the Akamai Foundation. The Clay Mathematics Institute (CMI) award, for a solution of outstanding elegance, and carrying a $3000 cash prize, was presented to Tiankai Liu for his solution to USAMO Problem 6. Two additional CMI awards, carrying a $1000 cash prize each, were presented to Anders Kaseorg and Matthew Tang for their solutions to USAMO Problem 5.

The top twelve students on the 2003 USAMO were (in alphabetical order):

Boris Alexeev	Cedar Shoals High School	Athens, GA
Jae Bae	Academy of Advancement in Science and Technology	Hackensack, NJ
Daniel Kane	West High School	Madison, WI
Anders Kaseorg	Charlotte Home Educators Association	Charlotte, NC
Mark Lipson	Lexington High School	Lexington, MA
Tiankai Liu	Phillips Exeter Academy	Exeter, NH
Po-Ru Loh	James Madison Memorial High School	Madison, WI
Po-Ling Loh	James Madison Memorial High School	Madison, WI
Aaron Pixton	Vestal Senior High School	Vestal, NY

Kwokfung Tang	Phillips Exeter Academy	Exeter, NH
Tony Zhang	Phillips Exeter Academy	Exeter, NH
Yan Zhang	Thomas Jefferson High School of Science and Technology	Alexandria, VA

The USA team members were chosen according to their combined performance on the 32nd annual USAMO and the Team Selection Test that took place at the Mathematical Olympiad Summer Program (MOSP) held at the University of Nebraska-Lincoln, June 15–July 5, 2003. Members of the USA team at the 2003 IMO (Tokyo, Japan) were Daniel Kane, Anders Kaseorg, Mark Lipson, Po-Ru Loh, Aaron Pixton, and Yan Zhang. Zuming Feng (Phillips Exeter Academy) and Gregory Galperin (Eastern Illinois University) served as team leader and deputy leader, respectively. The team was also accompanied by Melanie Wood (Princeton University) and Steven Dunbar (University of Nebraska-Lincoln), as the observer of the team leader and deputy leader, respectively.

At the 2003 IMO, gold medals were awarded to students scoring between 29 and 42 points, silver medals to students scoring between 19 and 28 points, and bronze medals to students scoring between 13 and 18 points. There were 37 gold medalists, 69 silver medalists, and 104 bronze medalists. There were three perfect papers (Fu from China, Le and Nguyen from Vietnam) on this very difficult exam. Loh's 36 tied for 12th place overall. The team's individual performances were as follows:

Kane	GOLD Medallist	Loh	GOLD Medallist
Kaseorg	GOLD Medallist	Pixton	GOLD Medallist
Lipson	SILVER Medallist	Y. Zhang	SILVER Medallist

In terms of total score (out of a maximum of 252), the highest ranking of the 82 participating teams were as follows:

Bulgaria	227	Romania	143
China	211	Turkey	133
USA	188	Japan	131
Vietnam	172	Hungary	128
Russia	167	United Kingdom	128
Korea	157	Canada	119
		Kazakhastan	119

The 2003 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Kiran Kedlaya, and Richard Stong. The Team Selection Test was prepared by Titu Andreescu and Zuming Feng. The MOSP was held at the University of Nebraska-Lincoln. Zuming Feng (Academic Director), Gregory Galperin, and Melanie Wood served as instructors, assisted by Po-Shen Loh and Reid Barton as junior instructors, and Ian Le and Ricky Liu as graders. Kiran Kedlaya served as guest instructor.

For more information about the USAMO or the MOSP, contact Steven Dunbar at `sdunbar@math.unl.edu`.

2 2002 Olympiad Results

Daniel Kane, Ricky Liu, Tiankai Liu, Po-Ru Loh, and Inna Zakharevich, all with perfect scores, tied for first on the USAMO. They shared college scholarships of $30000 provided by the Akamai Foundation. The Clay Mathematics Institute (CMI) award, for a solution of outstanding elegance, and carrying a $1000 cash prize, was presented to Michael Hamburg, for the second year in a row, for his solution to USAMO Problem 6.

The top twelve students on the 2002 USAMO were (in alphabetical order):

Steve Byrnes	Roxbury Latin School	West Roxbury, MA
Michael Hamburg	Saint Joseph High School	South Bend, IN
Neil Herriot	Palo Alto High School	Palo Alto, CA
Daniel Kane	West High School	Madison, WI
Anders Kaseorg	Charlotte Home Educators Association	Charlotte, NC
Ricky Liu	Newton South High School	Newton, MA
Tiankai Liu	Phillips Exeter Academy	Exeter, NH
Po-Ling Loh	James Madison Memorial High School	Madison, WI
Alison Miller	Home Educators Enrichment Group	Niskayuna, NY
Gregory Price	Thomas Jefferson High School of Science and Technology	Alexandria, VA
Tong-ke Xue	Hamilton High School	Chandler, AZ
Inna Zakharevich	Henry M. Gunn High School	Palo Alto, CA

The USA team members were chosen according to their combined performance on the 31st annual USAMO and the Team Selection Test that took place at the Mathematics Olympiad Summer Program (MOSP) held at the University of Nebraska-Lincoln, June 18–July 13, 2002. Members of the USA team at the 2002 IMO (Glasgow, United Kingdom) were Daniel

Kane, Anders Kaseorg, Ricky Liu, Tiankai Liu, Po-Ru Loh, and Tong-ke Xue. Titu Andreescu (Director of the American Mathematics Competitions) and Zuming Feng (Phillips Exeter Academy) served as team leader and deputy leader, respectively. The team was also accompanied by Reid Barton (Massachusetts Institute of Technology) and Steven Dunbar (University of Nebraska-Lincoln) as the observers of the team leader, and Zvezdelina Stankova (Mills College) as the observer of the deputy leader.

At the 2002 IMO, gold medals were awarded to students scoring between 29 and 42 points (there were three perfect papers on this very difficult exam), silver medals to students scoring between 23 and 28 points, and bronze medals to students scoring between 14 and 22 points. Loh's 36 tied for fourth place overall. The team's individual performances were as follows:

Kane	GOLD Medallist	T. Liu	GOLD Medallist
Kaseorg	SILVER Medallist	Loh	GOLD Medallist
R. Liu	GOLD Medallist	Xue	Honorable Mention

In terms of total score (out of a maximum of 252), the highest ranking of the 84 participating teams were as follows:

China	212	Taiwan	161
Russia	204	Romania	157
USA	171	India	156
Bulgaria	167	Germany	144
Vietnam	166	Iran	143
Korea	163	Canada	142

The 2002 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Gregory Galperin, Alexander Soifer, Richard Stong and Zvezdelina Stankova. The Team Selection Test was prepared by Titu Andreescu and Zuming Feng. The MOSP was held at the University of Nebraska-Lincoln. Because of a generous grant from the Akamai Foundation, the 2002 MOSP expanded from the usual 24–30 students to 176. An adequate number of instructors and assistants were appointed. Titu Andreescu (Director), Zuming Feng, Dorin Andrica, Bogdan Enescu, Chengde Feng, Gregory Galperin, Razvan Gelca, Alex Saltman, Zvezdelina Stankova, Walter Stromquist, Zoran Sunik, Ellen Veomett, and Stephen Wang served as instructors, assisted by Reid Barton, Gabriel Carroll, Luke Gustafson, Andrei Jorza, Ian Le, Po-Shen Loh, Mihai Manea, Shuang You, and Zhongtao Wu.

3 2001 Olympiad Results

The top twelve students on the 2001 USAMO were (in alphabetical order):

Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Luke Gustafson	Breckenridge, MN
Stephen Guo	Cupertino, CA
Daniel Kane	Madison, WI
Ian Le	Princeton Junction, NJ
Ricky I. Liu	Newton, MA
Tiankai Liu	Saratoga, CA
Po-Ru Loh	Madison, WI
Dong (David) Shin	West Orange, NJ
Oaz Nir	Saratoga, CA
Gregory Price	Falls Church, VA

Reid Barton was the winner of the Samuel Greitzer-Murray Klamkin award, given to the top scorer on the USAMO. Reid Barton, Gabriel D. Carroll, Tiankai Liu placed first, second, and third, respectively, on the USAMO. They were awarded college scholarships of $15000, $10000, $5000, respectively, by the Akamai Foundation. The Clay Mathematics Institute (CMI) award, for a solution of outstanding elegance, and carrying a $1000 cash prize, was presented to Michael Hamburg for his solution to USAMO Problem 6.

The USA team members were chosen according to their combined performance on the 30th annual USAMO and the Team Selection Test that took place at the MOSP held at the Georgetown University, June 5-July 3, 2001. Members of the USA team at the 2001 IMO (Washington, D.C., United States of America) were Reid Barton, Gabriel D. Carroll, Ian Le, Tiankai Liu, Oaz Nir, and David Shin. Titu Andreescu (Director of the American Mathematics Competitions) and Zuming Feng (Phillips Exeter Academy) served as team leader and deputy leader, respectively. The team was also accompanied by Zvezdelina Stankova (Mills College), as the observer of the team deputy leader.

At the 2001 IMO, gold medals were awarded to students scoring between 30 and 42 points (there were 4 perfect papers on this very difficult exam), silver medals to students scoring between 19 and 29 points, and bronze medals to students scoring between 11 and 18 points. Barton and Carroll both scored perfect papers. The team's individual performances were as follows:

Barton	Homeschooled	GOLD Medallist
Carroll	Oakland Technical HS	GOLD Medallist
Le	West Windsor-Plainsboro HS	GOLD Medallist
Liu	Phillips Exeter Academy	GOLD Medallist
Nir	Monta Vista HS	SILVER Medallist
Shin	West Orange HS	SILVER Medallist

In terms of total score (out of a maximum of 252), the highest ranking of the 83 participating teams were as follows:

China	225	India	148
USA	196	Ukraine	143
Russia	196	Taiwan	141
Bulgaria	185	Vietnam	139
Korea	185	Turkey	136
Kazakhstan	168	Belarus	135

The 2001 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Gregory Galperin, Alexander Soifer, Richard Stong and Zvezdelina Stankova. The Team Selection Test was prepared by Titu Andreescu and Zuming Feng. The MOSP was held at Georgetown University, Washington, D.C. Titu Andreescu (Director), Zuming Feng, Alex Saltman, and Zvezdelina Stankova served as instructors, assisted by George Lee, Melanie Wood, and Daniel Stronger.

4 2000 Olympiad Results

The top twelve students on the 2000 USAMO were (in alphabetical order):

David G. Arthur	Toronto, ON
Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Kamaldeep S. Gandhi	New York, NY
Ian Le	Princeton Junction, NJ
George Lee, Jr.	San Mateo, CA
Ricky I. Liu	Newton, MA
Po-Ru Loh	Madison, WI
Po-Shen Loh	Madison, WI
Oaz Nir	Saratoga, CA
Paul A. Valiant	Belmont, MA
Yian Zhang	Madison, WI

Reid Barton and Ricky Liu were the winners of the Samuel Greitzer-Murray Klamkin award, given to the top scorer(s) on the USAMO. The Clay Mathematics Institute (CMI) award was presented to Ricky Liu for his solution to USAMO Problem 3.

The USA team members were chosen according to their combined performance on the 29th annual USAMO and the Team Selection Test that took place at the MOSP held at the University of Nebraska-Lincoln, June 6–July 4, 2000. Members of the USA team at the 2000 IMO (Taejon, Republic of Korea) were Reid Barton, George Lee, Ricky Liu, Po-Ru Loh, Oaz Nir, and Paul Valiant. Titu Andreescu (Director of the American Mathematics Competitions) and Zuming Feng (Phillips Exeter Academy) served as team leader and deputy leader, respectively. The team was also accompanied by Dick Gibbs (Chair, Committee on the American Mathematics Competitions, Fort Lewis College), as the official observer of the team leader.

At the 2000 IMO, gold medals were awarded to students scoring between 30 and 42 points (there were four perfect papers on this very difficult exam), silver medals to students scoring between 20 and 29 points, and bronze medals to students scoring between 11 and 19 points. Barton's 39 tied for 5th. The team's individual performances were as follows:

Barton	Homeschooled	GOLD Medallist
Lee	Aragon HS	GOLD Medallist
Liu	Newton South HS	SILVER Medallist
P.-R. Loh	James Madison Memorial HS	SILVER Medallist
Nir	Monta Vista HS	GOLD Medallist
Valiant	Milton Academy	SILVER Medallist

In terms of total score (out of a maximum of 252), the highest ranking of the 82 participating teams were as follows:

China	218	Belarus	165
Russia	215	Taiwan	164
USA	184	Hungary	156
Korea	172	Iran	155
Bulgaria	169	Israel	139
Vietnam	169	Romania	139

The 2000 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Kiran Kedlaya, Alexander Soifer, Richard Stong and Zvez-delina Stankova. The Team Selection Test was prepared by Titu Andreescu and Kiran Kedlaya. The MOSP was held at the University of Nebraska-Lincoln.

Titu Andreescu (Director), Zuming Feng, Razvan Gelca, Kiran Kedlaya, Alex Saltman, and Zvezdelina Stankova served as instructors, assisted by Melanie Wood and Daniel Stronger.

5 1999 Olympiad Results

The top eight students on the 1999 USAMO were (in alphabetical order):

Reid W. Barton	Arlington, MA
Gabriel D. Carroll	Oakland, CA
Lawrence O. Detlor	New York, NY
Stephen E. Haas	Sunnyvale, CA
Po-Shen Loh	Madison, WI
Alexander B. Schwartz	Bryn Mawr, PA
Paul A. Valiant	Belmont, MA
Melanie E. Wood	Indianapolis, IN

Alexander (Sasha) Schwartz was the winner of the Samuel Greitzer-Murray Klamkin award, given to the top scorer on the USAMO. Newly introduced was the Clay Mathematics Institute (CMI) award, to be presented (at the discretion of the USAMO graders) for a solution of outstanding elegance, and carrying a $1000 cash prize. The CMI award was presented to Po-Ru Loh (Madison, WI; brother of Po-Shen Loh) for his solution to USAMO Problem 2.

Members of the USA team at the 1999 IMO (Bucharest, Romania) were Reid Barton, Gabriel Carroll, Lawrence Detlor, Po-Shen Loh, Paul Valiant, and Melanie Wood. Titu Andreescu (Director of the American Mathematics Competitions) and Kiran Kedlaya (Massachusetts Institute of Technology) served as team leader and deputy leader, respectively. The team was also accompanied by Walter Mientka (University of Nebraska, Lincoln), who served as secretary to the IMO Advisory Board and as the official observer of the team leader.

At the 1999 IMO, gold medals were awarded to students scoring between 28 and 39 points, silver medals to students scoring between 19 and 27 points, and bronze medals to students scoring between 12 and 18 points. Barton's 34 tied for 13th. The team's individual performances were as follows:

Barton	GOLD Medallist
Carroll	SILVER Medallist
Detlor	BRONZE Medallist

P.-S. Loh	SILVER Medallist
Valiant	GOLD Medallist
Wood	SILVER Medallist

In terms of total score, the highest ranking of the 81 participating teams were as follows:

China	182	Korea	164
Russia	182	Iran	159
Vietnam	177	Taiwan	153
Romania	173	USA	150
Bulgaria	170	Hungary	147
Belarus	167	Ukraine	136

The 1999 USAMO was prepared by Titu Andreescu (Chair), Zuming Feng, Kiran Kedlaya, Alexander Soifer and Zvezdelina Stankova. The MOSP was held at the University of Nebraska-Lincoln. Titu Andreescu (Director), Zuming Feng, Kiran Kedlaya, and Zvezdelina Stankova served as instructors, assisted by Andrei Gnepp and Daniel Stronger.

6 1999–2003 Cumulative IMO Results

In terms of total scores (out of a maximum of 1260 points for the last five years), the highest ranking of the participating IMO teams is as follows:

China	1048	Hungary	676
Russia	964	India	658
Bulgaria	918	Japan	658
USA	889	Ukraine	656
Korea	841	Turkey	624
Vietnam	823	Germany	603
Romania	741	Kazakhstan	583
Taiwan	733	Israel	563
Belarus	713	Canada	547
Iran	680	Australia	544

More and more countries now value the crucial role of meaningful problem solving in mathematics education. The competition is getting tougher and tougher. A top ten finish is no longer a given for the traditional powerhouses.

About the Authors

Titu Andreescu received his BA, MS, and PhD from the West University of Timisoara, Romania. The topic of his doctoral dissertation was "Researches on Diophantine Analysis and Applications." Titu teaches at the University of Wisconsin-Whitewater and is chairman of the USA Mathematical Olympiad. He served as director of the MAA American Mathematics Competitions (1998–2003), coach of the USA International Mathematical Olympiad Team (IMO) for 10 years (1993–2002), director of the Mathematical Olympiad Summer Program (1995–2002) and leader of the USA IMO Team (1995–2002). In 2002 Titu was elected member of the IMO Advisory Board, the governing body of the international competition. Titu received the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1994 and a "Certificate of Appreciation for his outstanding service as coach of the Mathematical Olympiad Summer Program in preparing the US team for its perfect performance in Hong Kong at the 1994 International Mathematical Olympiad" from the president of the MAA, in 1995.

Zuming Feng graduated with a PhD degree from Johns Hopkins University with emphasis on Algebraic Number Theory and Elliptic Curves. He teaches at Phillips Exeter Academy. He also serves as a coach (1997–2003)/deputy leader (2000–2002)/leader (2003) of the USA International Mathematical Olympiad (IMO) Team. Zuming is a member of the USA Mathematical Olympiad Committee (1999–2003). He was an assistant director (1999–2002) and later the academic director (2003) of the USA Mathematical Olympiad Summer Program (MOSP). Zuming received the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1996 and 2002.